Honors thesis presentation

Bowdoin College

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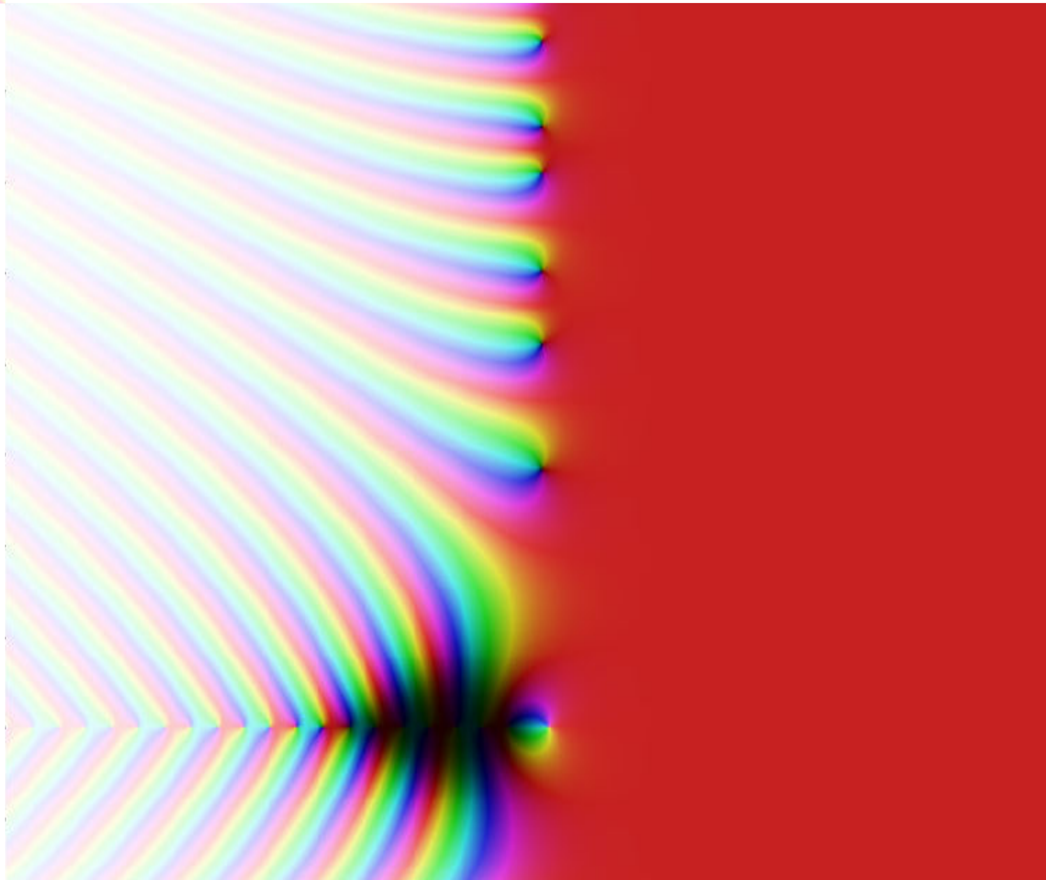
On the Dirichlet L-functions & the L-functions of cusp forms



L-functions

Any L-function has a Dirichlet series

$$L(s, X) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

where s and a_n are complex numbers and $\{a_n\}$ is determined by X.

e.g. The Riemann ζ-function

Dirichlet characters

The Dirichlet L-functions

Cusp forms

The L-functions of cusp forms

Elliptic curves

Number fields

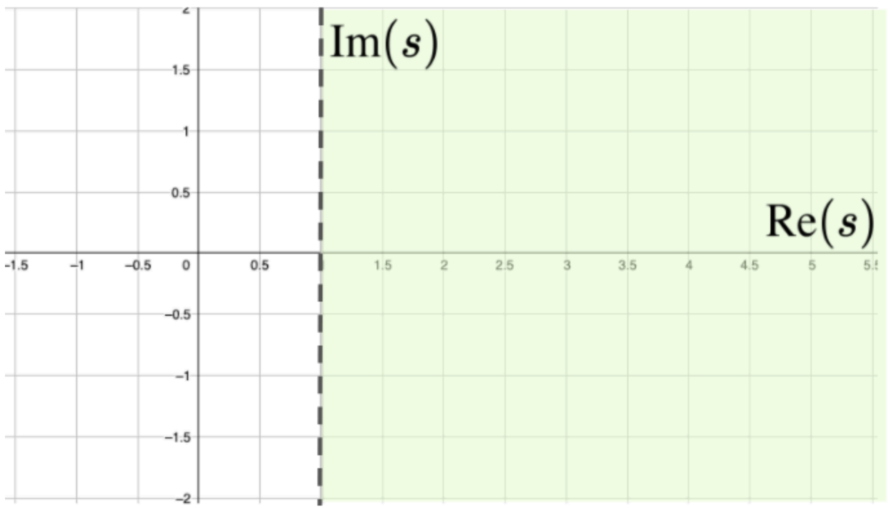
Euler product → Analytic continuation & Functional equation → Riemann hypothesis → Critical values

The Riemann ζ-function

Let s be a complex number.
The Riemann ζ-function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

which converges absolutely for $\mathrm{Re}(s) > 1$.

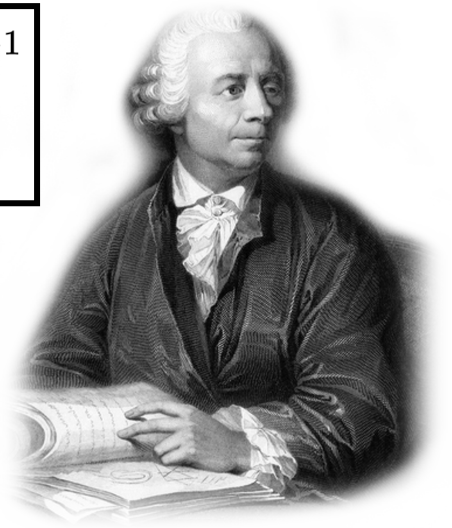


$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} + \cdots$$

$$\sum_{n=1}^{\infty} \frac{1}{n^0} = 1 + 1 + 1 + 1 + \cdots \quad ✖$$

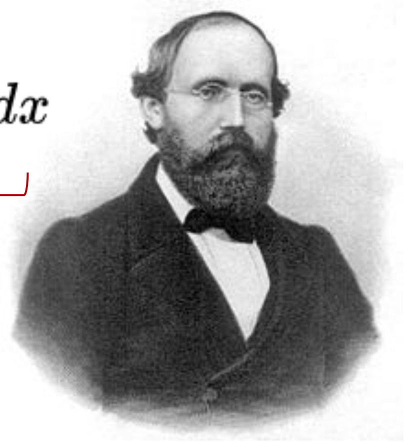
Euler product

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p} \left(1 - \frac{1}{p^s}\right)^{-1}}$$



Analytic continuation

For $s \in \mathbb{C}, s \neq 0, 1,$ $\quad \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \frac{1}{s-1} - \frac{1}{s} + \underbrace{\int_1^\infty \left(x^{\frac{s}{2}-1} + x^{\frac{-1-s}{2}}\right)\omega(x)\,dx}_{\text{uniformly convergent on every compact subset of the plane} \,\to\, \text{analytic}}$



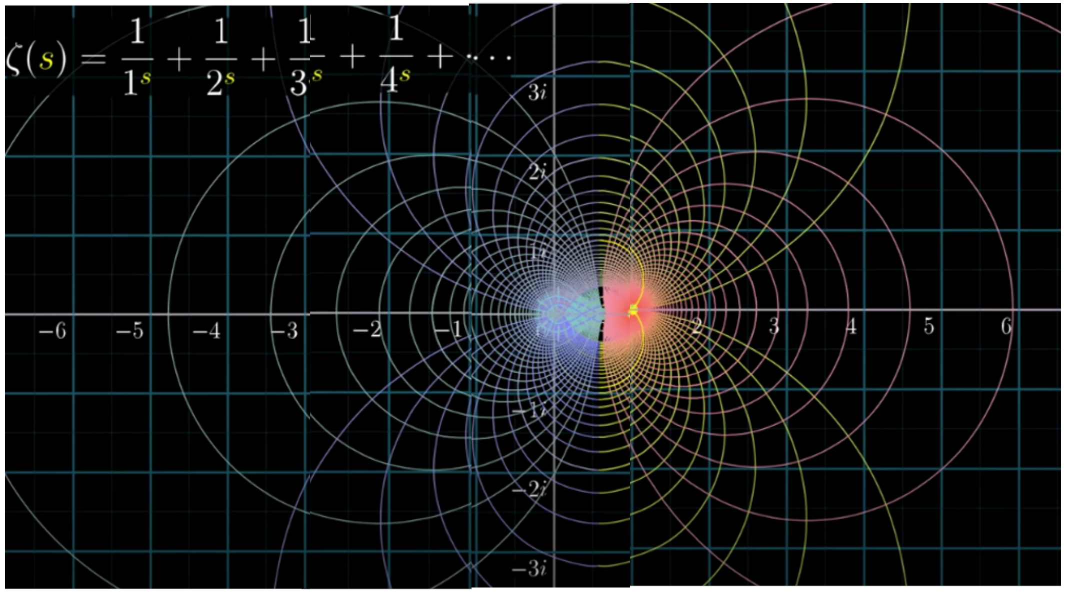
Functional equation

$$\pi^{-s/2}\Gamma\left(\tfrac{s}{2}\right)\zeta(s) \;=\; \pi^{(s-1)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) \qquad \Lambda(s) = \Lambda(1-s)$$

For $\mathrm{Re}(s) > 0$, the Euler Γ-function is defined by the integral formula

$$\Gamma(s) = \int_0^\infty e^{-t}t^s\,\frac{dt}{t}.$$

$$\omega(t) = \sum_{n=1}^\infty e^{-n^2\pi t}$$

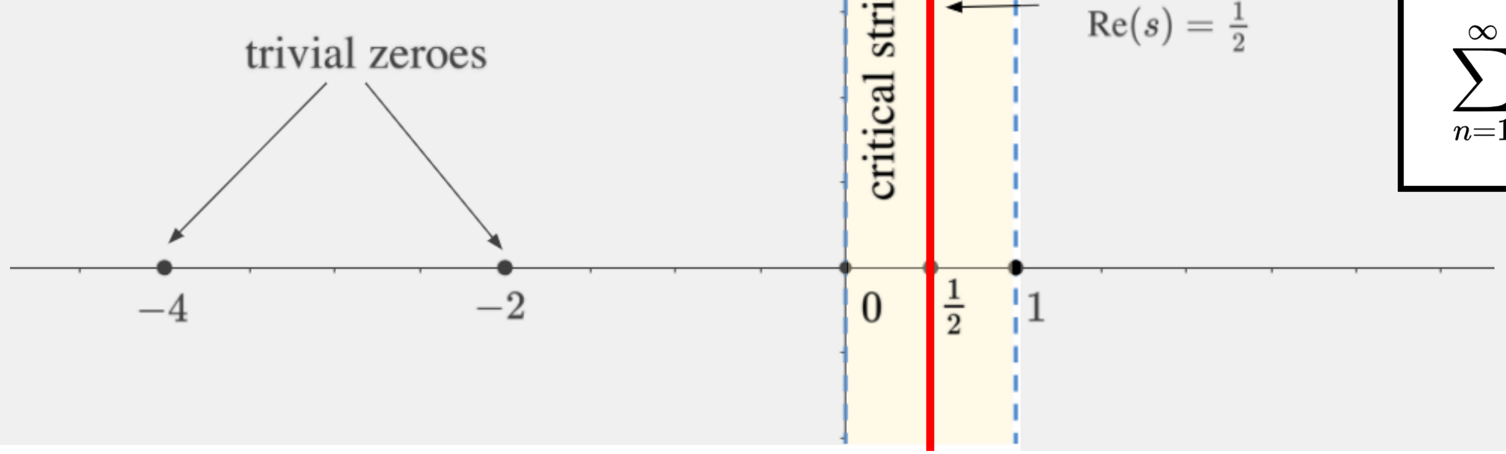


Riemann hypothesis

has simple poles at $s/2 = 0, -1, -2, -3, \ldots$ $\boxed{\infty \cdot 0}$ trivial zeroes at $s = -2, -4, -6, \ldots$

Analytic continuation: $\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \underbrace{\frac{1}{s-1} - \frac{1}{s} + \int_1^\infty \left(x^{\frac{s}{2}-1} + x^{\frac{-1-s}{2}}\right)\omega(x)\,dx}_{\text{analytic with two poles at } s=0,1}$

$$\boxed{\Lambda(s) = \Lambda(1-s)}$$



Euler product

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \in P} \frac{1}{\left(1 - \frac{1}{p^s}\right)}$$

Critical values

Functional equation $$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{(s-1)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

$$\boxed{\begin{array}{r}\text{has poles at } \frac{s}{2} = 0, -1, -2, -3, \ldots \\ s = 0, -2, -4, -6, \ldots\end{array}}$$

$$\boxed{\begin{array}{c}\text{has poles at } \frac{1-s}{2} = 0, -1, -2, -3, \ldots \\ s = 1, 3, 5, 7, \ldots\end{array}}$$

Critical values: $\zeta(m)$, where $m = \cancel{-1, -3, -5, \ldots}$ and $2, 4, 6, \ldots$

$$\zeta(2) = \sum_{n=1}^{\infty}\frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$$

$$\zeta(4) = \sum_{n=1}^{\infty}\frac{1}{n^4} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots = \frac{\pi^4}{90}$$

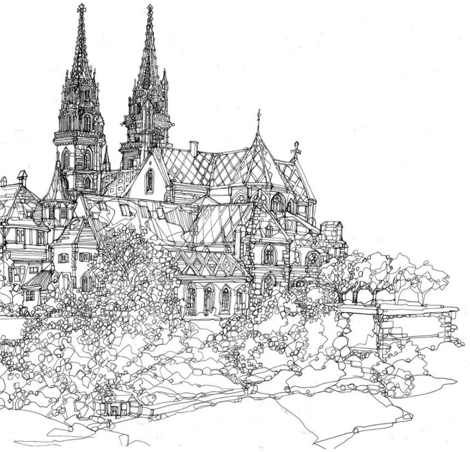
$$\zeta(6) = \sum_{n=1}^{\infty}\frac{1}{n^6} = 1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \cdots = \frac{\pi^6}{945}$$

$$\vdots$$

$$\boxed{\zeta(2k) = \frac{(-1)^{k-1}2^{2k-1}\pi^{2k}B_{2k}}{(2k)!}}$$

B_k is the k-th Bernoulli number

$$\sum_{k=0}^{\infty} B_k \frac{t^k}{k!} = \frac{t}{e^t - 1}$$



L-functions

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$$L(s, X) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

where s and a_n are complex numbers and $\{a_n\}$ is determined by X.

e.g. The Riemann ζ-function ☑

Dirichlet characters
The Dirichlet L-functions
?

Cusp forms
The L-functions of cusp forms

Euler product → Analytic continuation & Functional equation → Riemann hypothesis → Critical values

The Dirichlet-L functions

Given $\gcd(a, d) = 1$,
$\exists$ infinitely many primes in
$a,\ a + d,\ a + 2d,\ a + 3d,\ \ldots.$

Dirichlet characters

A function χ from $\mathbb{Z}$ to $\mathbb{C}$ is a Dirichlet character mod q if and only if it has the following properties:

(1) If $\gcd(a, q) > 1$ then $\chi(a) = 0$.
Otherwise, $\chi(a) \neq 0$;
(2) $\chi(ab) = \chi(a)\chi(b)$ for $a, b \in \mathbb{Z}$;
(3) $\chi(a) = \chi(b)$ if $a \equiv b \pmod{q}$.

Dirichlet L-functions

The Dirichlet L-function associated with χ can be defined as

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

which converges absolutely for $\mathrm{Re}(s) > 1$.

- $\chi_1(n) = 1$ for all $n \in \mathbb{Z}$ $\Rightarrow$ $L(s, \chi_1) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \cdots = \zeta(s)$

-

n	0	1	2	3
$\chi_4(n)$	0	1	0	-1

$\Rightarrow$ $L(s, \chi_4) = \frac{1}{1^s} - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \ldots$

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

Euler product

$$\boxed{\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}}$$

$\chi(-1) = 1$

n	0	1	2	3
$\chi'_4(n)$	0	1	0	1
$\chi_2(n)$	0	1	0	1

Analytic continuation

Functional equation

Let $q > 1$ and let χ be an even primitive character mod q.

$$2\left(\frac{\pi}{q}\right)^{-s/2} \Gamma\left(\frac{s}{2}\right) L(s,\chi) = \int_1^{\infty} \left(\frac{\tau(\chi)}{\sqrt{q}} \theta(\overline{\chi}, t) t^{\frac{1-s}{2}-1} + \theta(\chi, t) t^{s/2-1}\right) dt$$

$$\tau(\chi) = \sum_{a=1}^{q} \chi(a) e^{\frac{2\pi i a}{q}}$$

$$\left(\frac{\pi}{q}\right)^{-s/2} \Gamma\left(\frac{s}{2}\right) L(s,\chi) = \frac{\tau(\chi)}{\sqrt{q}} \left(\frac{\pi}{q}\right)^{(1-s)/2} \Gamma\left(\frac{1-s}{2}\right) L(1-s,\overline{\chi})$$

$$\Lambda_q(s,\chi) = \frac{\tau(\chi)}{\sqrt{q}} \Lambda_q(1-s,\overline{\chi})$$

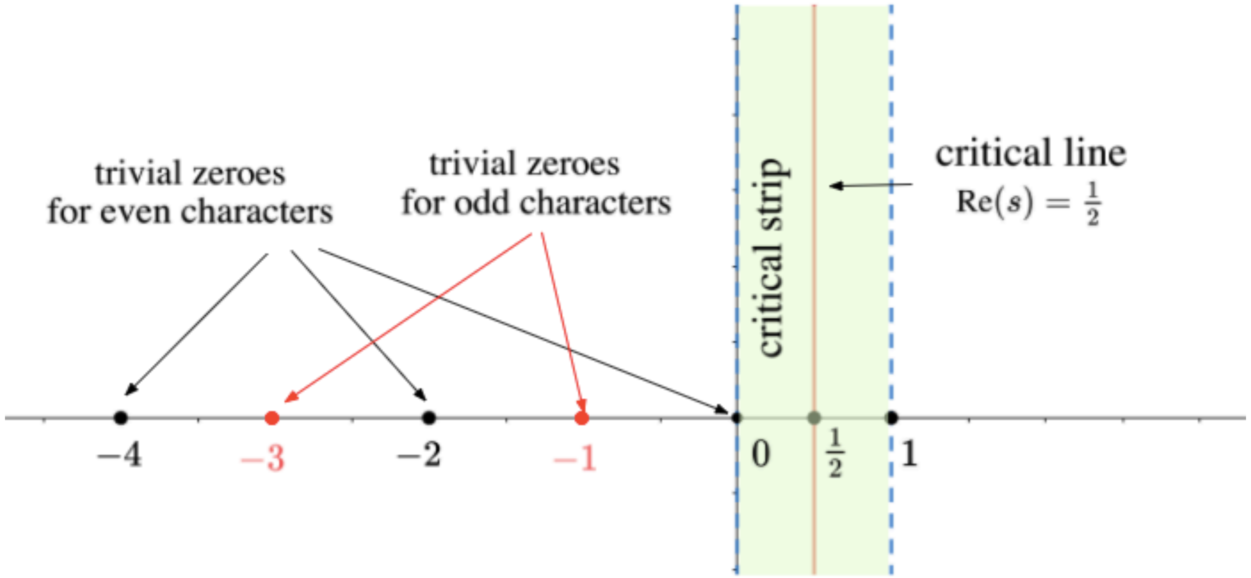
$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{(s-1)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

For an even primitive character χ, the trivial zeroes are at $s = 0, -2, -4, -6, \dots$ while the critical values are at $s = \cancel{-1, -3, -5, -7}, \dots$ and $2, 4, 6, 8, \dots$.

For an odd primitive character χ, the trivial zeroes are at $s = -1, -3, -5, -7, \dots$ while the critical values are at $s = \cancel{0, -2, -4, -6, -8}, \dots$ and $1, 3, 5, 7, \dots$.

Provided χ is not principal, let q be the conductor of χ and $p \in \{0, 1\}$ s.t. $\chi(-1) = (-1)^p \chi(1)$. For $k \equiv p \pmod 2$, $k \geq 1$, we have that

$$L(k, \chi) = (-1)^{1+(k-p)/2} \frac{\tau(\chi)}{2i^p} \left(\frac{2\pi}{q}\right)^k \frac{B_{k,\overline{\chi}}}{k!}$$



Generalized Riemann hypothesis

An alternative approach to computing $L(2k+1,\chi_4)$

n	0	1	2	3
$\chi_4(n)$	0	1	0	-1

$$L(2k+1,\chi_4) = \frac{1}{1^{2k+1}} - \frac{1}{3^{2k+1}} + \frac{1}{5^{2k+1}} - \frac{1}{7^{2k+1}} + \dots = \beta(2k+1)$$

1 Goal: $\beta(2k+1) = (-1)^{k+1}\left(\frac{\pi}{2}\right)^{2k+1}\frac{B_{2k+1,\chi_4}}{(2k+1)!}$

2 Auxiliary function:

$$I(k,m) = \int_0^{1/2} E_{2k}(t)\sin((2m+1)\pi t)\,dt, \quad \text{for integers } k,m \geq 0.$$

3 Properties of $E_k(t)$ and recurrence relation

$$\sum_{m=0}^{\infty}(-1)^m I(k,m) = \frac{(-1)^k(2k)!}{\pi^{2k+1}}\beta(2k+1).$$

4 Telescoping series and trig identities

$$\sum_{m=0}^{\infty}(-1)^m I(k,m) = -\frac{B_{2k+1,\chi}}{(2k+1)2^{2k+1}}$$

Euler polynomials

$$\sum_{k=0}^{\infty} E_k(x)\frac{t^k}{k!} = \frac{2e^{xt}}{e^t+1},$$

where $|t| \leq \pi, x \in \mathbb{R}$.

☑ Submitted for publication

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The theory of modular forms

The modular group

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

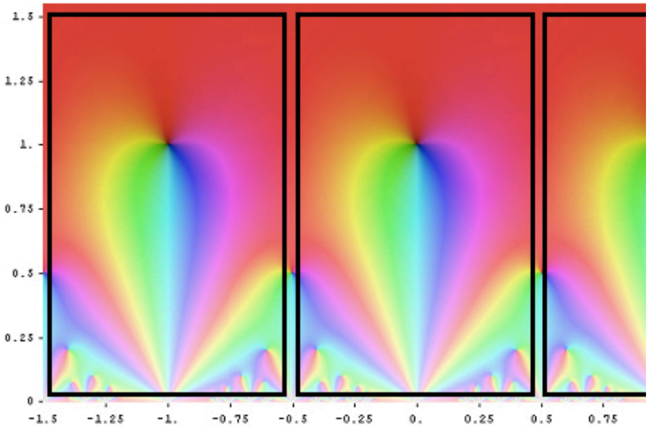
Action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ (upper half-plane)

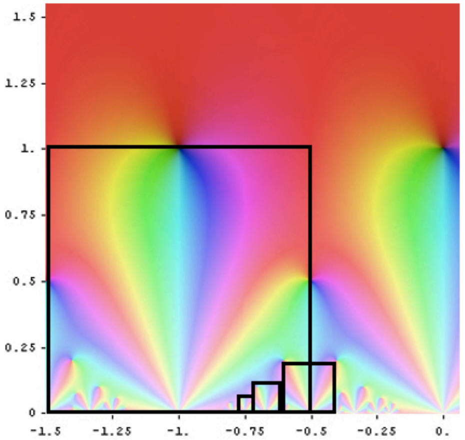
For any $z \in \mathbb{H}$ and any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$$

defines an action of $\mathrm{SL}_2(\mathbb{Z})$.

"The Möbius transformation"





Modular forms for $\mathrm{SL}_2(\mathbb{Z})$

A modular form f is a holomorphic function $\mathbb{H} \to \mathbb{C}$ s.t.

1. Modularity condition:

$$f(\gamma z) = (cz+d)^k f(z) \text{ for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

1*. $f(z+1) = f(z)$ and $f(-1/z) = z^k f(z)$.

This k is called the weight of the modular form f.

2. f is holomorphic at infinity.

$f(z)$ has q-series expansion

$$f(q) = \sum_{n=-\infty}^{\infty} a_n q^n, \text{ where } q = e^{2\pi i z}.$$

If the first n s.t. $a_n \neq 0$ is ≥ 0, f is holomorphic at infinity.

Cusp forms: A modular form with $a_0 = 0$

Examples of modular forms

Modular form of an odd weight?

The zero function

- ☑ Holomorphic on $\mathbb{H}$
- ☑ Modularity condition
- ☑ Holomorphic at infinity

Eisenstein series

For $z \in \mathbb{H}$ and even integer $k \geq 4$,

$$E_k(z) = \frac{1}{2\zeta(k)} \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(mz+n)^k}$$

defines the Eisenstein series of weight k.

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

➡ modular form, where $\sigma_s(n) = \sum_{r \mid n} r^s$.

Cusp forms:

A modular form f with $f(q) = \sum_{n=1}^{\infty} a_n q^n$

Ramanujan's cusp form

$$\Delta(z) = \frac{E_4(z)^3 - E_6(z)^2}{1728}$$

L-functions of cusp forms

Let f be a cusp form of weight k,

given by a q-expansion $f = \sum_{n=1}^{\infty} a_n q^n$.

Then, $L(s, f) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$,

which converges absolutely for all s

such that $\mathrm{Re}(s) > k/2 + 1$.

Analytic continuation

$$L(s, f)(2\pi)^{-s}\Gamma(s) = \int_1^{\infty} f(ix)\left(i^k x^{k-s} + x^s\right)\frac{dx}{x}$$

Functional equation

$$(2\pi)^{-s}\Gamma(s)L(s, f) = i^k (2\pi)^{-(k-s)}\Gamma(k-s)L(k-s, f)$$

$$\Lambda(s, f) = i^k \Lambda(k-s, f)$$

Critical values at $s = 1, 2, \ldots, k-1$

$$\widetilde{\Lambda}(s, f) = i^k \widetilde{\Lambda}(1-s, f) \quad \text{(scaled)}$$

Euler product

If a cusp form f of weight k is a normalized eigenform, then

$$L(s, f) = \prod_p \frac{1}{1 - a_p p^{-s} + p^{k-1-2s}}.$$

Grand Riemann hypothesis

For a scaled cusp form f that is a normalized eigenform.

Euler product	Analytic continuation & Functional equation	Riemann hypothesis	Critical values

Conclusion

e.g. The Riemann ζ-function

Dirichlet characters

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Selberg class (S)

★ Euler product

★ Analytic continuation

★ Functional equation

★ Ramanujan's conjecture

For every $\varepsilon > 0$, $a_n = O(n^{\varepsilon})$

Riemann hypothesis

For any L-function in S, the non-trivial zeroes of f lie on $\mathrm{Re}(s) = 1/2$

References

[1] T. Arakawa, T. Ibukiyama, M. Kaneko, and D. Zagier. *Bernoulli Numbers and Zeta Functions.* Springer, 2014.

[2] A. Balog. An Introduction to Analytic Number Theory, 2020.

[3] M. R. Murty, M. Dewar, and H. Graves. *Problems in the Theory of Modular Forms.* Springer, 2015.

[4] J. Costa, L. Salamatian, and A. Ferreira Costa. Visualising Modular Forms, 2016.

[5] J Evertse. Chapter 4: Characters and Gauss Sums, 2013.

[6] N. Tanabe and N. Wattanawanichkul. An Alternative Approach to Computing $\beta(2k+1)$ (submitted for publication).

[7] J. L. Taylor. *Complex Variables.* American Mathematical Society, 2011.

[8] L. Zheng. A Concise Survey of the Selberg Class of L-function, 2011.