A New Approach to Computing $\beta(2k+1)$

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Introduction

The Dirichlet L-functions:

$$L(s,\chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \frac{\chi(1)}{1^s} + \frac{\chi(2)}{2^s} + \frac{\chi(3)}{3^s} + \cdots$$

where s is any complex number whose real part is greater than 1.

Special cases:

When χ is the trivial character, i.e., the sequence 1, 1, 1, 1, ….

The Riemann zeta function

$$\zeta(2k) := \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{1}{1^{2k}} + \frac{1}{2^{2k}} + \frac{1}{3^{2k}} + \cdots$$

When χ is the primitive character mod 4, i.e., the sequence 1, 0, -1, 0, ….

The Dirichlet beta function

$$\beta(2k+1) := \sum_{n=0}^{\infty} \frac{(-1)^k}{(2n+1)^{2k+1}} = \frac{1}{1^{2k+1}} - \frac{1}{3^{2k+1}} + \frac{1}{5^{2k+1}} + \cdots$$

This is the function that we are interested in ☺

Objects and their relations

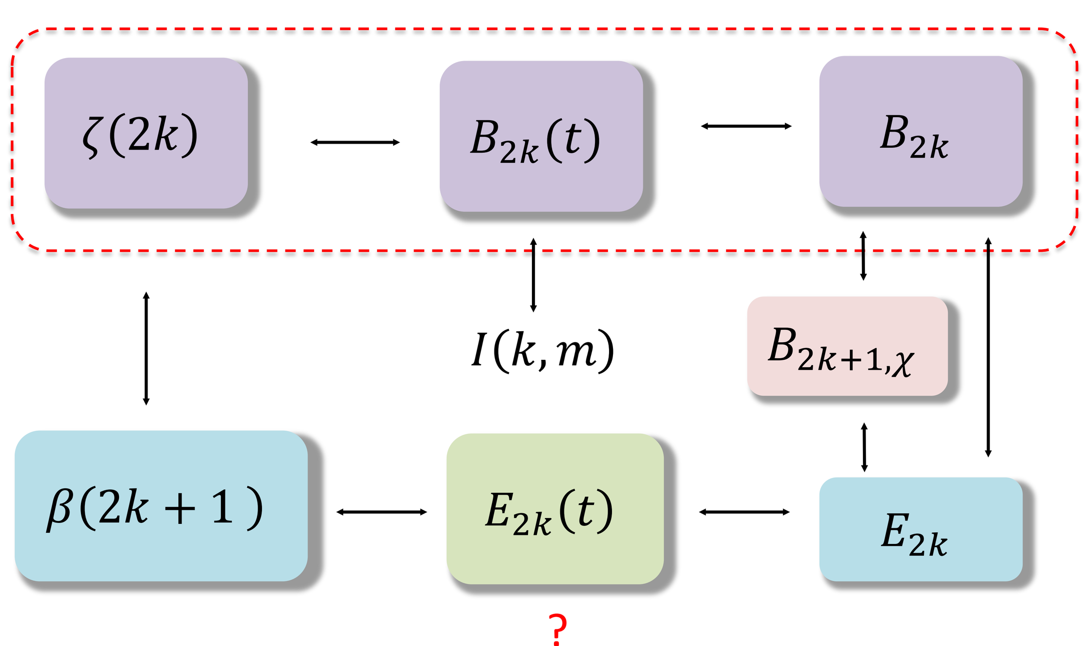

Diagram1: Relations among the objects in our research

Bernoulli numbers

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}$$

e.g. $B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, \ldots$

Bernoulli polynomials

$$\frac{te^x}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}$$

e.g. $B_0(x) = 1, B_1(x) = x - \frac{1}{2}, B_3(x) = x^2 - x + \frac{1}{6}, \ldots$

Euler numbers

$$\frac{e^x}{e^{2x} + 1} = \sum_{n=0}^{\infty} E_n \frac{x^n}{n!}$$

e.g. $E_1 = 0, E_2 = -1, E_3 = 0, E_4 = 5, \ldots$

Euler polynomials

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}$$

e.g. $E_0(x) = 1, E_1(x) = x - \frac{1}{2}, E_3(x) = x^2 - x, \ldots$

Generalized Bernoulli numbers

$$\sum_{a=1}^{f} \chi(a) \frac{te^{at}}{e^{ft} - 1} = \sum_{n=0}^{\infty} B_{k,\chi} \frac{t^k}{k!}$$

Connection to Euler numbers:

$$\frac{E_{2k}}{2} = -\frac{B_{2k+1,\chi_4}}{2k+1}$$

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Methods

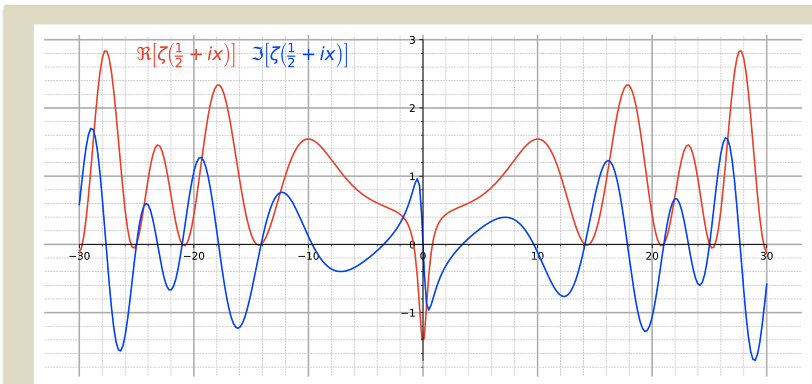
	Ciaurri et al in 2015	Our research
Functions	$\zeta(2k) := \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{1}{1^{2k}} + \frac{1}{2^{2k}} + \frac{1}{3^{2k}} + \cdots$	$\beta(2k+1) := \sum_{n=0}^{\infty} \frac{(-1)^k}{(2n+1)^{2k+1}} = \frac{1}{1^{2k+1}} - \frac{1}{3^{2k+1}} + \frac{1}{5^{2k+1}} + \cdots$
Closed forms	$\zeta(2k) = \frac{(-1)^{k-1}2^{2k-1}\pi^{2k}}{(2k)!} B_{2k}$	$\beta(2k+1) = \frac{(-1)^{k+1}\pi^{2k+1}}{2^{2k+2}(2k)!} E_{2k}$
Tools	A. Properties of Bernoulli Polynomials B. Telescopic sums of Trigonometric functions	I. Properties of Euler Polynomials II. Telescopic sums of Trigonometric functions
Auxiliary function	$I(k,m) = \int_0^1 B_{2k}(t)\cos(2m\pi t)\,dt, k \geq 0, m \geq 1$	$J(k,m) = \int_0^{1/2} E_{2k}(t)\sin((2m+1)\pi t)\,dt, k \geq 0, m \geq 1$
Modified Auxiliary function	$B_{2k}(t) - B_{2k}$ → $I^*(k,m)$	$E_{2k}(t) - E_{2k}\sin(t)$ → $J^*(k,m)$
Analysis	$I(k,m)$ ~ A: $\frac{(-1)^{k-1}(2k)!}{2^{2k}\pi^{2k}}\zeta(2k)$ = $I^*(k,m)$ ~ B: $\frac{B_{2k}}{2}$	$J(k,m)$ ~ I: $\frac{(-1)^k(2k)!}{\pi^{2k+1}}\beta(2k+1)$ = $J^*(k,m)$ ~ II: $-\frac{E_{2k}}{2^{2k+2}}$

Research Goals & Questions

- To find a new proof for evaluating the special values of the Dirichlet L-function for the primitive character mod 4, also known as the Dirichlet beta function, $\boldsymbol{\beta(2k+1)}$, by resembling the idea in [2].
- To understand why the idea in [2] could not be simply adapted to a computation of $\boldsymbol{L(2k+1,\chi)}$, for any arbitrary Dirichlet character χ.

FUTURE RESEARCH: $\beta(2k)$ and $L(2k+1,\chi)$.

Why is it important?



The real part (red) and imaginary part (blue) of the Riemann zeta function along the critical line Re(s) = ½

- The distribution of prime numbers
- Modern Security
- Statistics/physics

References

[1] T. Arakawa, T. Ibukiyama, and M. Kaneko, *Bernoulli numbers and Zeta- Functions.* Springer, Tokyo, 2014.
[2] Ó. Ciaurri, L. M. Navas, F. J. Ruiz, and J. L. Varona, *A simple computation of* $\zeta(2k)$ The American Mathematical Monthly 122, No. 5, 2015.
[3] J. Neukirch, *Algebraic Number Theory.* Translated edition of Algebraische Zahlentheorie. Springer, Berlin, Heidelberg, 1999.
[4] H. M. Srivastava and Á. Pintér, *Remarks on some relationships between the Bernoulli and Euler polynomials.* Applied Mathematics Letters, Volume 17, Issue 4, April 2004, p. 375-380.