# RATIONAL EQUIVALENCE AND VARIETY ISOMORPHISM

PLOY WATTANAWANICHKUL

Abstract. This paper is a continuation of what we have learned in MATH 3702, Algebraic Geometry. This work closely follows Chapter 5 of [1] while the first four chapters of the book have already been covered in class. In this paper, we want to explore the notion of variety isomorphism, meaning to understand when two varieties are "the same." The first three sections lay down the necessary background. Section 4 then gives us a method to determine variety isomorphism through the isomorphism of their coordinate rings, while Section 5 offers another way using the rational equivalence of their function fields.

## Contents



## 1. Polynomial Mappings

In class, we have learned about affine varieties, ideals, and the correspondence between these two objects. However, here, we present another noteworthy concept, a mapping between two varieties. The notion of a mapping between two things of the same kind is a common and useful tool in study abstract algebra, so we may be familiar to some ideas presented here.

**Definition 1.1** (Polynomial mappings)**.** *Let $V \subseteq k^m, W \subseteq k^n$ be varieties. A function $\phi : V \to W$ is said to be a polynomial mapping if there exists polynomials $f_1, \ldots, f_n \in k[x_1, \ldots, x_m]$ such that*

$$\phi(a_1, \ldots, a_m) = (f_1(a_1, \ldots, a_m), \ldots, f_n(a_1, \ldots, a_m))$$

*for all $(a_1, \ldots, a_m) \in V$. We say that the $n$-tuple of polynomials $(f_1, \ldots, f_n) \in (k[x_1, \ldots, x_m])^n$ represents $\phi$. The $f_i$ are the components of this representation.*

There are two kinds of maps that we have discussed in class and are indeed polynomial mappings: **parametrization** and **projection**.

**Example 1.2** (Parametrization)**.** *We know that the twisted cubic can be parametrized as* $x = t + u, y = t^2 + 2tu$, *and* $z = t^3 + 3t^2u$. *We can then write this parametrization as a map*

$$\phi(t, u) = (t + u, t^2 + tu, t^3 + 3t^2u).$$

*From* [1, Ch. 3, § 2], *we know that, given the domain of* $\phi$ *is* $V = \mathbb{R}^2$, *the image of this map, which is the twisted cubic, can be defined by the affine variety*

$$W = \boldsymbol{V}(x^3z - (3/4)x^2y^2 - (3/2)xyz + y^3 + (1/4)z^2).$$

*Hence,* $\phi : V \to W$ *is a polynomial mapping.*

We also note that given a polynomial parametrization, we can always find the defining equation or the implicit representation of the smallest variety containing the parametrization (See [1, Ch. 3, Pg. 133-135]).

**Example 1.3** (Projection)**.** *Consider the projection mappings* $\pi_1 : \mathbb{C}^n \to \mathbb{C}^{n-1}$ *defined by*

$$\pi_l(a_1, \dots, a_n) = (a_{l+1}, \dots, a_n).$$

*By the elimination theory (see* [1, Ch. 3, §2]*), given an ideal* $V = \boldsymbol{V}(I) \subset \mathbb{C}^n$, *we know that* $W = V(I_l)$ *will be the smallest variety containing* $\pi_l(V)$, *where* $I_l = I \cap \mathbb{C}[x_{l+1}, \dots, x_n]$.

A more concrete example of a polynomial mapping is as follows:

**Exercise 1.4** (§1, ex.1)**.** *Given* $V = \boldsymbol{V}(y - x^2, z - x^3) \subseteq \mathbb{R}^3$ *be a twisted cubic and* $W = \boldsymbol{V}(v - u - u^2) \subseteq \mathbb{R}^2$. *Then,* $\phi(x, y, z) = (xy, z + x^2y^2)$ *is a polynomial mapping from* $V$ *to* $W$.

*Proof.* Consider

$$(z + x^2y^2) - (xy) - (xy)^2 = z - xy.$$

Since $y - x^2$ and $z - x^3$ vanishes for any $(x, y, z) \in V$, we have that $y = x^2$ and $z = x^3$. Then, $xy = x^3 = z$, i.e., $z - xy = 0$. Therefore, $(z + x^2y^2) - (xy) - (xy)^2 = 0$, meaning for $(x, y, z) \in V$, $\phi(x, y, z) \in W$. Note that an easier way to show this problem is to use the parametrization of the twisted cubic. □

However, we are not jumping into studying a general mapping $\phi : V \to k^n$ but, instead, considering a map $\phi : V \to k$, which is a scalar polynomial function. It turns out that finding such $f_1, f_2, \dots, f_n$ that defines the desired mapping is not a key issue, but finding a unique one is. Thus, the following proposition is needed to define the sameness of two varieties:

**Proposition 1.5.** *Let* $V \subseteq k^m$ *be an affine variety. Then:*

(1) *$f$ and $g \in k[x_1, \dots, x_m]$ represent the same polynomial function on $V$ if and only if $f - g \in \boldsymbol{I}(V)$.*
(2) *$(f_1, \dots, f_n)$ and $(g_1, \dots, g_n)$ represent the same polynomial mapping from $V$ to $k^n$ if and only if $f_i - g_i \in \boldsymbol{I}(V)$ for each $i, 1 \leq i \leq n$.*

Note that, for a special case when $V = k^m$, we then have $\mathbf{I}(V) = \{0\}$, i.e., $f$ and $g$ represent the same polynomial function on $V$ if $f = g$. The following exercise proves this special case:

**Exercise 1.6** (§1, ex.7)**.** *Show that if $k$ is an infinite field and $V \subseteq k^m$ is a variety, then $\boldsymbol{I}(V) = \{0\}$ if and only if $V = k^m$.*

*Proof.* ($\subseteq$): Proof by contradiction. Suppose that $\mathbf{I}(V) = \{0\}$ and $V \neq k^m$. There exists $(x_1, \dots, x_m)$ such that $f(x_1, x_2, \dots, x_m) \neq 0$, where $f$ is the zero function, which is impossible.

($\supseteq$) This direction which is done by induction on the number of variables $m$. The base case is clear because a non-zero single variable polynomial $f$ of degree $n$ can have at most $n$ distinct roots. Since $k$ is infinite, $f$ must be identically zero.

Now assume that the converse is true for $m - 1$ and let $f \in k[x_1, x_2, \dots, x_m]$ such that $f$ vanishes at all point of $k^m$. We can write

$$f = \sum_{i=0}^{N} g_i(x_1, \dots, x_{m-1}) x_m^i,$$

where $g_i \in k[x_1, x_2, \dots, x_m]$. For any fixed $(a_1, \dots, a_{m-1}) \in k^{m-1}$, we have $f(a_1, \dots, a_{m-1}, x_m)$ vanishes for all $x_m \in k$. Since $f(a_1, \dots, a_{m-1}, x_m)$ is a single variable polynomial, it must be the zero function. This means $g_i(x_1, \dots, x_{m-1}) = 0$ for any $(a_1, \dots, a_{m-1})$. Thus, by the inductive hypothesis, $g_i$ must be the zero function, so does $f$. □

To allow us to further study scalar polynomial mappings, let us define the following notation.

**Definition 1.7.** *We denote by $k[V]$ the collection of polynomial functions $\phi : V \to k$.*

Since $k$ is a field, it can be shown that $k[V]$ is indeed a "commutative ring." Furthermore, some geometric information of $V$ can be described using $k[V]$.

**Proposition 1.8.** *Let $V \subseteq k^n$ be an affine variety. The following are equivalent:*
(1) *$V$ is irreducible.*
(2) *$\boldsymbol{I}(V)$ is a prime ideal.*
(3) *$k[V]$ is an integral domain.*

Another information we can retrieve using polynomial mappings is "isomorphism" of varieties. Given two varieties, if there is a 1-1, onto, polynomial mapping from $V$ to $\mathbb{C}$, with a polynomial inverse, then the two varieties are isomorphic (colloquially, they are the same).

In conclusion, the structure of $k[V]$ informs us of some geometric properties of the variety $V$. Later, we will see that it leads us to the concept of "classification," in which we group isomorphic varieties.

## 2. Quotients of Polynomial Rings

This section explores the notion of quotients of polynomial rings. Our motivation here is that $k[V]$ is indeed a special case of the quotient of $k[x_1, \dots, x_n]$ modulo an ideal $I$.

**Definition 2.1** (Congruence modulo $I$). *Let $I \subseteq k[x_1, \dots, x_n]$ be an ideal, and let $f, g \in k[x_1, \dots, x_n]$. We say $f$ and $g$ are congruent modulo $I$, written $f \equiv g \pmod{I}$, if $f - g \in I$.*

In fact, the congruence modulo $I$ is an equivalence relation on $k[x_1, \dots, x_n]$, which means it is reflexive, symmetric, and transitive. In particular, given $I = \mathbf{I}(V)$ an ideal of the variety $V$, we have that $f \equiv g \pmod{I}$ if and only if $f$ and $g$ define the same function on V, according to Proposition 1.5. In other words, we obtain the following proposition.

**Proposition 2.2.** *The distinct polynomial functions $\phi : V \to k$ are in one-to-one correspondence with the equivalence classes of polynomials under congruence modulo $I(V)$.*

Now, we can define quotients of polynomial rings.

**Definition 2.3** (Quotients). *The quotient of $k[x_1, \dots, x_n]$ modulo $I$, written $k[x_1, \dots, x_n]/I$, is the set of equivalence classes for congruence modulo $I$:*

$$k[x_1, \dots, x_n]/I = \{[f] \,|\, f \in k[x_1, \dots, x_n]\}.$$

For example, consider $k = \mathbb{R}, n = 1$, and $I = \langle x^2 - 2 \rangle$. By the division algorithm, any element in $\mathbb{R}[x]$ belongs to one of the equivalence classes of the form $[ax + b]$, where $a, b \in \mathbb{R}$. In other words, $\mathbb{R}[x]/I = \{[ax + b] \,|\, a, b \in \mathbb{R}\}$. Let us now define some operations on $\mathbb{R}[x]/I$.

**Proposition 2.4** (Operations). *The following operations are well-defined on classes $\mathbb{R}[x]/I$ :*

$$[f] + [g] = [f + g] \quad (\textit{sum in } k[x_1, \dots, x_n]),$$

$$[f] \cdot [g] = [f \cdot g] \quad (\textit{product in } k[x_1, \dots, x_n]).$$

The above proposition means that for any $f' \in [f]$ and $g' \in [g]$, $[f' + g'] = [f + g]$ and $[f' \cdot g'] = [f \cdot g]$. Using the well-defined operations in Proposition 2.4, we can show that associativity and commutativity of addition and multiplication and the distributive law follow. Moreover, we also have the additive identity $[0]$ and the multiplicative identity $[1]$ in the quotient $k[x_1, \dots, x_n]/I$. In conclusion, the quotient $k[x_1, \dots, x_n]/I$ is a "commutative ring" under the sum and product operations given Proposition 2.4.

Now that we have both $k[V]$ and $k[x_1, \dots, x_n]/\mathbf{I}(V)$ are commutative rings, we can further show that these two rings are in fact "the same," in the following sense:

**Theorem 2.5** (Isomorphism)**.** *The one-to-one correspondence between the elements of $k[V]$ and the elements of $k[x_1, \dots, x_n]/\boldsymbol{I}(V)$ given in Proposition 2.2 preserves sums and products. To put it in an abstract algebra term, these two rings are "isomorphic."*

*Proof.* Let $\Phi : k[x_1, \dots, x_n]/\mathbf{I}(V) \to k[V]$ be the mapping $\Phi([f]) = \phi$, where $\phi$ is the polynomial function represented by $f$. The one-to-one correspondence can be shown by using Proposition 2.2. For the ring homomorphism part (i.e., $\Phi$ preserves the sum and product), we let $f$ represents $\phi$ and $g$ represents $\psi$. Then, we have

$$\Phi([f] + [g]) = \Phi([f + g]) = \phi + \psi = \Phi([f]) + \Phi([g]),$$

where the product can be proved similarly. □

Usually, we write $R \cong S$ to denote that $R$ and $S$ are isomorphic, so, in this case, we have $k[x_1, \dots, x_n]/\mathbf{I}(V) \cong k[V]$. A related exercise is provided below:

**Exercise 2.6** (§2, ex.6)**.** *Show that $\mathbb{R}[x]/\langle x^2 - 4x + 3\rangle$ is not an integral domain.*

*Proof.* Take $V = \{x \mid x^2 - 4x + 3 = 0\}$ and $k = \mathbb{R}$. From Theorem 2.5, we know that $k[V] \cong \mathbb{R}[x]/\langle x^2 - 4x + 3\rangle$. Moreover, we know that $V$ is reducible because we can write it as $V = V_1 \cup V_2$, where $V_1$ is defined by $x - 1$ and $V_2$ by $x - 3$. By Proposition 1.8, $k[V]$ is not an integral domain, so $V = \{x \mid x^2 - 4x + 3 = 0\}$ is also not an integral domain. □

**A natural question to ask**: since we know that for any $V$, there are many $I$ apart from $\mathbf{I}(V)$ such that $V = \mathbf{V}(I)$, do we always have $k[x_1, \dots, x_n]/I \cong k[V]$? The answer is no, and one reason has to do with the "nilpotent" elements in a commutative ring. (Given a commutative ring $R$, if $a^n = 0$ for some $n \geq 1$, then $a$ is a nilpotent.)

To explain, we compare two quotients $k[x_1, \dots, x_n]/I$ and $k[x_1, \dots, x_n]/\mathbf{I}(V)$ for another ideal $I$ with $\mathbf{V}(I) = V$. If $I$ is not radical, then $\sqrt{I} \neq I$, meaning there is an element $f \in \sqrt{I} \setminus I$. Thus, in $k[x_1, \dots, x_n]/I$, there is an element $[f] \neq 0$ but $[f]^n = 0$ for some $n$, i.e., there is a nilpotent element in this quotient. However, as $\mathbf{I}(V)$ is always radical, there will not be any nilpotents. Thus, $k[x_1, \dots, x_n]/I \ncong k[x_1, \dots, x_n]/\mathbf{I}(V)$.

Let us also look at the following exercise about the nilpotents in a commutative ring:

**Exercise 2.7** (§2, ex.11)**.** *Let $R$ be a commutative ring. Show that the set of nilpotent elements of $R$ forms an ideal in $R$.*

*Proof.* Let $I$ be a set of all nilpotents in $R$. We first note that $0 \in I$ because $0^n = 0$ for any $n$.

Now, suppose that $a, b \in I$, where $a^n = b^m = 0$. If we expand $(a+b)^{m+n-1}$,the for each term either the degree of $a \geq m$ or the degree of $b \geq n$. This is because if the degree of $a \leq m-1$ and $b \leq n-1$, the total degree would be less than $m+n-2$. Hence, every term of $(a+b)^{m+n-1}$ vanishes, meaning $(a+b)^{m+n-1} = 0$. In other words, $a+b \in I$. (Note that commutativity allows us to write every term in the form $C_{ij}a^ib^j$.)

Lastly, for any $a \in I$ where $a^m = 0$ and any $r \in R$, we have $(ra)^m = r^m a^m = 0$ by commutativity. Hence, $ra \in I$. $\square$

As we have seen, things can be more complicated when we consider an arbitrary ideal instead of the ideal of a variety. However, there is a tight relation between ideals in the quotients $k[x_1, \ldots, x_n]/I$ and in $k[x_1, \ldots, x_n]$, which will lead us to more understanding of $k[x_1, \ldots, x_n]/I$.

**Proposition 2.8.** *Let $I$ be an ideal in $k[x_1, \ldots, x_n]$. The ideals in the quotient $k[x_1, \ldots, x_n]/I$ are in one-to-one correspondence with the ideals of $k[x_1, \ldots, x_n]$ containing $I$ (i.e., the ideals $J$ such that $I \subseteq J \subseteq k[x_1, \ldots, x_n]$.*

Particularly, the one-to-one correspondence can be given as follows:

$$\begin{array}{rcl} \{J \mid I \subseteq J \subseteq k[x_1, \ldots, x_n]\} & & \{\widetilde{J} \subseteq k[x_1, \ldots, x_n]/I\} \\ J & \rightarrow & J/I = \{[j] \mid j \in J\} \\ J = \{j \mid [j] \in \widetilde{J}\} & \leftarrow & \widetilde{J}. \end{array} \tag{2.1}$$

Note that each of the two maps can be shown to be the inverse of each other. A nice corollary of this proposition is that every ideal $\widetilde{J}$ in the quotient ring $k[x_1, \ldots, x_n]/I$ is finitely generated. This is because we can apply the Hilbert Basis Theorem to the ideal $J$ corresponding to $\widetilde{J}$.

## 3. Algorithmic Computations in $k[x_1, \ldots, x_n]/I$

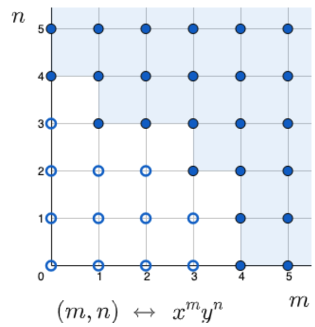
In this section, we exploit the division algorithm to find simple representatives of equivalence classes for congruence modulo $I$, where $I$ is an ideal in $k[x_1, \ldots, x_n]$.

**Proposition 3.1.** *Fix a monomial ordering on $k[x_1, \ldots, x_n]$ and let $I \subseteq k[x_1, \ldots, x_n]$ be an ideal. Also, we let $\langle LT(I) \rangle$ denote the ideal generated by the leading terms of elements of $I$.*

1. *Every $f \in k[x_1, \ldots, x_n]$ is congruent modulo $I$ to a unique polynomial $r$ which is a $k$-linear combination of the monomials in the complement of $\langle LT(I) \rangle$.*
2. *The elements of $\{x^\alpha \mid x^\alpha \notin \langle LT(I) \rangle\}$ are "linearly independent modulo $I$," meaning that if we have*

$$\sum_\alpha c_\alpha x^\alpha \equiv 0 \mod I,$$

*where the $x^\alpha$ are all in the complement of $\langle LT(I) \rangle$, then $c_\alpha = 0$ for all $\alpha$.*




Let us look at a related example. Given $I = \langle xy^3 - x^2, x^3y^2 - y\rangle$ in $\mathbb{R}[x, y]$, using graded lex order gives us a Gröbner basis $G = \{x^3y^2 - y, x^4 - y^2, xy^3 - x^2, y^4 - xy\}$ for $I$. Thus, $\langle \mathrm{LT}(I)\rangle = \langle x^3y^2, x^4, xy^3, y^4\rangle$. The diagram below shows that the 12 monomials, namely, $1, x, x^2, x^3, y, xy, x^2y, x^3y, y^2, xy^2, x^2y^2, y^3$ do not belong to $\langle \mathrm{LT}(I)\rangle$. Hence, by Proposition 3.1, the remainder of any $f \in \mathbb{R}[x, y]$ divided by $G$ is a linear combination of these monomials.

Note that if we use another monomial ordering, we may get a different Gröbner basis of $I$, say $G'$. Hence, we also get a different set of monomials that span the set of $\overline{f}^{G'}$, for all $f \in \mathbb{R}[x, y]$. However, given fixed $I$, we always get the same number of such monomials, i.e., 12 in this case. The next proposition tells us why this should be true.

**Proposition 3.2.** *Let $I \subseteq k[x_1, \ldots, x_n]$ be an ideal. Then, $k[x_1, \ldots, x_n]/I$ is isomorphic as a $k$-vector space to $S = Span(x^\alpha \,|\, x^\alpha \notin \langle LT(I)\rangle)$.*

To prove this, first note that Proposition 3.1 tells us that $\Phi : k[x_1, \ldots, x_n]/I \to S$, defined by $\Phi([f]) = \overline{f}^G$, is a one-to-one correspondence. Then, by further showing that $\Phi$ preserves vector space operations, we obtain the above propostion.

This proposition explains why we always have the same number of monomials that define $k[x_1, \ldots, x_n]/I$. This is because the quotient $k[x_1, \ldots, x_n]/I$ must be isomorphic to some $k$-vector space regardless of which monomial ordering we use to find a Gröbner basis.

Furthermore, we also have a nice way to handle addition and multiplication in $k[x_1, \ldots, x_n]/I$.

**Proposition 3.3.** *Let $I$ be an ideal in $k[x_1, \ldots, x_n]/I$ and let $G$ be a Gröbner basis of $I$ with resepect to any monomial order. For each $[f] \in k[x_1, \ldots, x_n]/I$, we get the standard representative $\overline{f} = \overline{f}^G$ in $S = Span(x^\alpha \,|\, x^\alpha \notin \langle LT(I)\rangle)$. Then:*

(1) *$[f] + [g]$ is represented by $\overline{f} + \overline{g}$.*
(2) *$[f] \cdot [g]$ is represented by $\overline{\overline{f} \cdot \overline{g}}$.*

Back to the example we have discussed earlier, we have observed that $x^4$ and $y^4$ are in $\langle \mathrm{LT}(I)\rangle \in \mathbb{R}[x, y]$, and hence we have that the set $S = \mathrm{Span}(x^\alpha \,|\, x^\alpha \notin \langle \mathrm{LT}(I)\rangle)$ is finite.

Following from such observation, the next theorem gives us a criterion to determine when a variety in $\mathbb{C}^n$ is finite.

**Theorem 3.4** (Finiteness Theorem). *Let $I \subseteq k[x_1, \ldots, x_n]$ be an ideal and fix a monomial ordering on $k[x_1, \ldots, x_n]$. Consider the following five statements:*

1. *For each $i, 1 \leq i \leq n$, there is some $m_i \geq 0$ such that $x_i^{m_i} \in \langle LT(I) \rangle$.*
2. *Let $G$ be a Gröbner basis for $I$. Then for each $i, 1 \leq i \leq n$, there is some $m_i \geq 0$ such that $x_i^{m_i} = LM(g)$ for some $g \in G$.*
3. *The set $\{x^\alpha \mid x^\alpha \notin \langle LT(I) \rangle\}$ is finite.*
4. *The $k$-vector space $k[x_1, \ldots, x_n]/I$ is finite-dimensional.*
5. *$\boldsymbol{V}(I) \subseteq k^n$ is a finite set.*

*Then $(1) - (4)$ are equivalent and they all imply $(5)$. Furthermore, if $k$ is algebraically closed, then $(1) - (5)$ are all equivalent.*

Given an algebraically closed field $k$, the above theorem gives us a way to determine if $\mathbf{V}(I) \subseteq k^n$ is a finite set (or a "zero-dimensional" variety) by looking at the $k$-vector space $k[x_1, \ldots, x_n]/I$. Even better, we can even give a "bound" for the dimension as a vector space of $k[x_1, \ldots, x_n]/I$ through the following proposition. This bound also implies the number of points of $V(I)$ in the case that $k$ is algebraically closed.

**Proposition 3.5.** *Let $I \subseteq k[x_1, \ldots, x_n]$ be an ideal such that for each $i$, some power $x_i^{m_i} \in \langle LT(I) \rangle$ and set $V = \boldsymbol{V}(I)$. Then:*

1. *The number of points of $V$ is at most dim $k[x_1, \ldots, x_n]/I$ (where "dim" means dimension as a vector space over $k$).*
2. *The number of points of $V$ is at most $m_1 \cdot m_2 \cdots m_n$.*
3. *If $I$ is radical and $k$ is algebraically closed, then equality holds in part $(i)$, i.e., the number of points in $V$ is exactly dim $k[x_1, \ldots, x_n]/I$.*

In conclusion, Theorem 3.4 tells us the finiteness of solutions of $\mathbf{V}(I)$ and Proposition 3.5 gives us the number of solutions. In fact, by using the elimination and extension theorems we learned in class, we can actually "find" all the possible solution by working backward. To understand how to do so, let us illustrate through the following exercises:

**Exercise 3.6** (§3, ex.5). *Let $I = \langle y + x^2 - 1, xy - 2y^2 + 2y \rangle \subseteq \mathbb{R}[x, y]$. Compute $\boldsymbol{V}(I)$ and compare the two bounds we get from Proposition 3.5 part (1) and part (2).*

*Proof.* We compute a Gröbner basis for $I$ using lex order with $x > y$, and get

$$G = \{x^2 + y - 1, xy - 2y^2 + 2y, y^3 - (7/4)y^2 + (3/4)y\}.$$

Thus, $\langle \mathrm{LT}(I)\rangle = \langle x^2, xy, y^3\rangle$. Then, by Proposition 3.2, $\{1, x, y, y^2\}$ forms a basis for the vector space of remainders modulo $I$.

Let us now compute $\mathbf{V}(I)$. First, consider $I_1 = G \cap \mathbb{R}[y] = \{y^3-(7/4)y^2+(3/4)y\}$. Then, we get that the partial solutions in $\mathbf{V}(I_{i-1})$ are $a = 0, 3/4$, and 1 because $y^3 - (7/4)y^2 + (3/4)y = y(y - 3/4)(y - 1)$. Then, we consider the whole $G$. Notice that there is a polynomial with $x^2$ as leading term, so the extension theorem implies that the partial solutions extends to $(x, a) = (x, 0), (x, 3/4)$, and $(x, 1)$.

To find possible $x$, we let $g_1 = x^2+y-1$, $g_2 = xy-2y^2+2y$, $g_3 = y^3-(7/4)y^2+(3/4)y$. Then, the desired $x$'s are the solutions of the equation system $g_1(x, a) = g_2(x, a) = g_3(x, a) = 0$ for all $a$. We have 3 possible cases:

(1) $a = 0$: We have $g_1(x, a) = x^2 - 1 = 0$ while the other two functions vanish, so $x = \pm 1$.
(2) $a = 3/4$: We have $g_1(x, a) = x^2 - 1/4 = 0$ and $g_2(x, a) = 3x/4 - 3/8 = 0$, so $x = 1/2$.
(3) $a = 1$: We have $g_1(x, a) = x^2 = 0$ and $g_2(x, a) = x = 0$, so $x = 0$.

Therefore, $(x, y) = (1, 0), (-1, 0), (1/2, 3/4)$, and $(0, 1)$ are the elements of $\mathbf{V}(I)$.

Lastly, by Proposition 3.5 part (2), since we have $x^2$ and $y^3$ in $\langle \mathrm{LT}(I)\rangle$, there are at most $2 \cdot 3 = 6$ points in $V$. However, using part (1) of Proposition 3.5, we know that the number of points of $V$ is at most $\dim k[x_1, \dots, x_n]/I$ which is 4. Thus, part (1) gives a better bound. □

**Exercise 3.7** (§3, ex.6). *Let* $V = \boldsymbol{V}(x_3 - x_1^2, x_4 - x_1x_2, x_2x_4 - x_1x_5, x_4^2 - x_3x_5) \subseteq \mathbb{C}^5$. *Using any monomial order, determine a collection of monomials spanning the space of remainders modulo a Gröbner basis for the ideal generated by the equations of* $V$. *Is* $V$ *a finite set?*

*Proof.* We compute a Gröbner basis using lex oreder with $x_1 > x_2 > x_3 > x_4 > x_5$, and get

$$G = \{x_1^2 - x_3, x_1x_2 - x_4, x_1x_4 - x_2x_2, x_1x_5 - x_2x_4, x_2^2x_3 - x_4,$$
$$x_2^2x_4 - x_4x_5, x_2x_3x_4^3 - x_2x_3x_4, x_3x_5 - x_4^2, x_4^4 - x_4^2\}.$$

Only $x_1$ and $x_4$ have $m_1 = 2$ and $m_4 = 4$ such that $x_1^2, x_4^4 \in \langle \mathrm{LT}(I)\rangle$. Since we are working in $\mathbb{C}$ which is algebraically closed, Theorem 3.4 (1) $\leftrightarrow$ (5) implies that $V$ is not finite. □

Hence, we have shown a way to determine if a variety is finite and to exploit the theories we have learned in class to solve equations in the zero-dimensional case, i.e. when a variety has finite elements.

## 4. The Coordinate Ring of an Affine Variety

Using the tools we have developed in the previous sections, we now study the ring $k[V]$ which is isomorphic to the quotient $k[x_1, \dots, x_n]/\mathbf{I}(V)$. We note that for $V \subseteq k^n$, each $[x_i] : V \to k$ is a polynomial function. Thus, just like how $x_1, x_2, \dots, x_n$ can be viewed as coordinates of the ring of polynomials $k[x_1, \dots, x_n]$, we can also view $[x_1], [x_2], \dots, [x_n]$ as coordinates of polynomials functions in $k[V]$. In other words, each polynomial function on $V$ is a $k$-linear combination of products of $[x_i]$.

**Definition 4.1.** *The coordinate ring of an affine variety $V \subseteq k^n$ is the ring $k[V]$.*

We want to study the algebra-geometry dictionary like we have done in Chapter 4, but now we change $k^n \to V$ and $k[x_1, \ldots, x_n] \to k[V]$. Some necessary definitions are as follows:

**Definition 4.2.** *Let $V \subseteq k^n$ be an affine variety.*

(1) *For any ideal $J = \langle \phi_1, \ldots, \phi_n \rangle \subseteq k[V]$, we define a "subvariety" $\boldsymbol{V}_V(J)$ of $V$ as*

$$\boldsymbol{V}_V(J) = \{(a_1, \ldots, a_n) \in V \mid \phi(a_1, \ldots, a_n) = 0 \text{ for all } \phi \in J\}.$$

(2) *For each subset $W \subset V$, we define*

$$\boldsymbol{I}_V(W) = \{\phi \in k[V] \mid \phi(a_1, \ldots, a_n) = 0 \text{ for all } (a_1, \ldots, a_n) \in W\}.$$

Like how we relate varieties in $k^n$ to ideals in $k[x_1, \ldots, x_n]$, we now want to explore how (sub)varieties in $V$ are related to ideals in $k[V]$.

**Proposition 4.3.** *Let $V \subseteq k^n$ be an affine variety.*

(1) *For each ideal $J \subseteq k[V], W = \boldsymbol{V}_V(J)$ is an affine variety variety in $k^n$ contained in $V$.*
(2) *For each subset $W \subseteq V$, $\boldsymbol{I}_V(W)$ is an ideal of $k[V]$.*
(3) *If $J \subseteq k[V]$ is an ideal, then $J \subseteq \sqrt{J} \subseteq \boldsymbol{I}_V(\boldsymbol{V}_V(W))$.*
(4) *If $W \subseteq V$ is a subvariety, then $W = \boldsymbol{V}_V(\boldsymbol{I}_V(W))$.*

Let us illustrate the above definitions and propositions through the following exercise.

**Exercise 4.4.** *Let $C$ be the twisted cubic curve in $k^3$. Show that $C$ is a subvariety of the surface $S = \boldsymbol{V}(xz - y^2)$ and find an ideal $J \subseteq k[S]$ such that $C = \boldsymbol{V}_S(J)$.*

*Proof.* Consider $(x, y, z) \in k^3$ such that $y - x^2$ and $z - x^3$ vanish. We know that such points are of the form $(t, t^2, t^3)$, where $t \in k$ Since $xz - y^2 = tt^3 - (t^2)^2 = 0$, this means $C \subseteq S$.

Since we know the polynomials $y - x^2$ and $z - x^3$ define $C$, it is easy to see that the ideal $J$ such that $C = \mathbf{V}_S(J)$ is $\langle [x^3 - z], [x^2 - y] \rangle$. We also note that, from Proposition 4.3 (4), we know that $C \subseteq S$, so $C = \mathbf{V}_S(\mathbf{I}_S(C))$. Clearly, $\mathbf{I}_S(C) \subseteq k[S]$, hence our $J$ is $\mathbf{I}_S(C)$. □

The following theorem then highlight some properties of the ideal-variety correspondence.

**Theorem 4.5.** *Let $k$ be an algebraically closed field and let $V \subseteq k^n$ be an affine variety.*

(1) *("The Nullstellensatz in $k[V]$") If $J$ is any ideal in $k[V]$, then*

$$\boldsymbol{I}_V(\boldsymbol{V}_V(J)) = \sqrt{J} = \{[f] \in k[V] \mid [f]^m \in J\}.$$

(2) *The correspondences below are inclusion-reversing bijections and are mutual inverses:*

$$\begin{array}{ccc} \text{affine varities} & \xrightarrow{\mathbf{I}_V} & \text{radical ideals} \\ W \subseteq V & \xleftarrow{\mathbf{V}_V} & J \subseteq k[V] \end{array}$$

(3) *Under the above correspondence, points of $V$ correspond to maximal ideals of $k[V]$.*

We can notice all the statements above are the analogs of what we have seen in [1, Ch.4], but here we just change $\mathbf{I}$ and $\mathbf{V}$ to $\mathbf{I}_V$ and $\mathbf{V}_V$.

Although we briefly mentioned variety isomorphism in §2, let us formally define it here.

**Definition 4.6.** *Let $V \subseteq k^m$ and $W \subseteq k^n$ be affine varieties. We say that $V$ and $W$ are "isomorphic" if there exist polynomial mappings $\alpha : V \to W$ and $\beta : W \to V$ such that $\alpha \circ \beta = id_W$ and $\beta \circ \alpha = id_V$.*

One way to exploit the variety isomorphism is to determine if there is a parametrization of a variety. For example, suppose we have $W \subseteq k^n$ and $V = k^m$ isomorphic. Then there must be a 1-1 onto polynomial mapping from $k^m \to W$, which is indeed a parametrization of $W$.

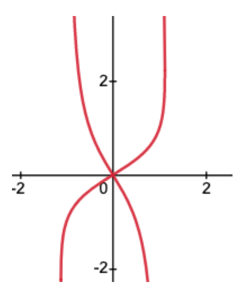
Let us provide an example of how we use variety isomorphism in problem solving. In this problem, we want to find a parametrization of the intersection between

$$Q_1 = \mathbf{V}(x^2 - xy - y^2 + z^2) = \mathbf{V}(f_1) \quad \text{and} \quad Q_2 = \mathbf{V}(x^2 - y^2 + z^2 - z) = \mathbf{V}(f_2).$$

We know that $C = \mathbf{V}(f_1, f_1 + cf_2)$ for any $c \in \mathbb{R} \setminus \{0\}$, so we can use $c$ such that $f_1 + cf_2$ is simple. We also denote the surface $\mathbf{V}(f_1 + cf_2)$, which contain $C$, by $F_C$. It turns out that $Q = F_{-1} = \mathbf{V}(z - xy)$ is a variety that is much easier to understand because $Q \cong \mathbb{R}^2$. To clarify, we consider the following polynomial mappings:

$$\alpha : \mathbb{R}^2 \to Q, (x, y) \mapsto (x, y, xy) \quad \text{and} \quad \pi : Q \to \mathbb{R}^2, (x, y, z) \mapsto (x, y).$$

It can be shown easily that $\alpha \circ \pi = \mathrm{id}_Q$ and $\pi \circ \alpha = \mathrm{id}_{\mathbb{R}^2}$. These mappings then tell us that we can understand $C$ by finding a parametrization of $\pi(C)$ and retrieve that of $C$ using $\alpha$. Since $C = \mathbf{V}(x^2 - xy - y^2 + z^2, z - xy)$, we can plug in $z = xy$ in the former defining equation and obtain that $\pi(C)$ is defined by $x^2 - xy - y^2 + x^2y^2 = 0$. Using this equation, we can obtain a parametrization for $\pi(C)$ and finally for $C$.




In general, if we consider $f(x, y) \in k[x, y]$ and the graph $V = \mathbf{V}(z - f(x, y)) \subseteq k^3$. Just like how we said that $\mathbf{V}(z - xy) \cong \mathbb{R}^2$, the curve $\mathbf{V}(z - f(x, y))$ is indeed always isomorphic to $k^2$. This is because the projection on the $(x, y)$-plane $\pi : V \to k^2$ and the parametrization of the graph $\alpha : k^2 \to V, \alpha(x, y) = (x, y, f(x, y))$ are inverse mappings.

We can see that one of the challenges in the previous example is that how to determine if two varieties are isomorphic. To answer this question, we first consider the following proposition.

**Proposition 4.7.** *Let $V$ and $W$ be varieties.*

(1) *Let $\alpha : V \to W$ be a polynomial mapping. Then for every polynomial function $\phi : W \to k$, the composition $\phi \circ \alpha : V \to k$ is also a polynomial function. Furthermore, we define the "pullback mapping" $\alpha^* : k[W] \to k[V]$ by $\alpha^*(\phi) = \phi \circ \alpha$. This is a ring homomorphism which is the identity on the constant functions $k \subseteq k[W]$. (meaning that $\alpha^*([a]) = [a]$ for any constant function $[a]$ on $W$.)*
(2) *Conversely, let $\Phi : k[W] \to k[V]$ be a ring homomorphism which is the identity on constants. Then there is a unique polynomial mapping $\alpha : V \to W$ such that $\Phi = \alpha^*$.*

This proposition leads us to the following theorem which gives us a nice way to determine variety isomorphism through their coordinate rings.

**Theorem 4.8.** *Two affine varieties $V \subseteq k^m$ and $W \subseteq k^n$ are isomorphic if and only if there is an isomorphism $k[V] \cong k[W]$ of coordinate rings which is identity on constant functions.*

In conclusion, variety isomorphism can help us understand some complicated curves like the example of curve $C$ in this section. Moreover, in general, given two isomorphic varieties, both should share important properties, for example, dimension and irreducibility. The following exercise affirms our intuition about the irreduciblity of two isomorphic varieties.

**Exercise 4.9** (§4, ex.17)**.** *Let $\phi : V \to W$ be an isomorphism of affine varieties. Prove that $V$ is irreducible if and only if $W$ is irreducible.*

*Proof.* First we want to show that if $f : R \to S$ is a ring isomorphism, then $R$ is an integral domain if and only if $S$ is an integral domain. Note that the ring isomorphism $\phi : R \to S$ maps the additive identity in $R$ to the one in $S$ because $\phi(0_R + 0_R) = \phi(0_R) + \phi(0_R)$. Let us assume that $R$ is an integral domain, so for $x, y \in R$ whenever $xy = 0_R$, either $x$ or $y$ must be $0_R$. Then, $0_S = \phi(xy) = \phi(x)\phi(y)$ and either $\phi(x)$ or $\phi(y)$ is 0.

Now, to prove the statement, we assume that $V$ is irreducible. Since $V \cong W$, the coordinate rings $k[V]$ and $k[W]$ must be isomorphic by Theorem 4.8. We know from Proposition 1.8 that $V$ is irreducible implies that $k[V]$ is an integral domain. Since $k[V] \cong k[W]$, we then have $k[W]$ is also an integral domain, so $W$ is irreducible, again, by Proposition 1.8. □

The following section will generalize the idea of polynomial functions. In particular, instead of focusing on the ring of polynomial functions $k[V]$, we will work on the *field* of *rational functions*, which we denote as $k(V)$.

## 5. Rational Functions on a Variety

From MATH 2702, we know that if $R$ is an integral domain, then we can form the "field of fractions" of $R$, which we denote as $FF(R)$. The elements in $FF(R)$ are just "fractions" $r/s$ where $r, s \in R, s \neq 0$. Addition and multiplication can be defined just like those for the rational numbers, namely,

$$\frac{r}{s} + \frac{r'}{s'} = \frac{rs' + r's}{ss'} \quad \text{and} \quad \frac{r}{s} \cdot \frac{r'}{s'} = \frac{rr'}{ss'}.$$

Thus, it is clear why $R$ must be integral domain since we do not want to have the denominators of any product or sum equal 0. We note that this field $FF(R)$ also contains the ring $R$ itself.

With the idea discussed above, given $V \subseteq k^n$ is irreducible, we know that $k[V]$ must be an integral domain (see Proposition 1.8). Then, we can define $FF(k[V])$ as follows:

**Definition 5.1.** *Let $V$ be an irreducible affine variety in $k^n$. We call $FF(k[V])$ the function field (or field of rational functions) of $k[V]$, and we denote it as $k(V)$. More explicitly,*

$$k(V) \;=\; \{\phi/\psi \mid \phi, \psi \in k[V], \psi \neq 0\} \;=\; \{[f]/[g] \mid f, g \in k[x_1, \ldots, x_n], g \notin \boldsymbol{I}(V)\}.$$

Similar to the term polynomial mappings, rational mappings can be defined as follows:

**Definition 5.2.** *Let $V \subseteq k^m$ and $W \subseteq k^n$ be irreducible affine varieties. A rational mapping form $V$ to $W$ is a function represented by*

$$\phi(x_1, \ldots, x_m) = \left(\frac{f_1(x_1, \ldots, x_m)}{g_1(x_1, \ldots, x_m)}, \ldots, \frac{f_n(x_1, \ldots, x_m)}{g_n(x_1, \ldots, x_m)}\right),$$

*where $f_i/g_i \in k(x_1, \ldots, x_m)$ satisfy:*

1. *$\phi$ is defined at some point of $V$.*
2. *For every $(a_1, \ldots, a_m) \in V$ where $\phi$ is defined, $\phi(a_1, \ldots, a_m) \in W$.*

It is clear that because of the denominators of a rational mapping, we need to be careful in defining the equality of two rational mappings and also for the composition. With this reason, the rest definitions and propositions in this section will be more or less the same as the ones we have seen before, with some further restriction about the domain of each mapping. Note that we will use the symbol "$V \dashrightarrow W$" to indicate that not every element in $V$ is well-defined.

**Definition 5.3.** *Two rational mappings $\phi, \psi : V \dashrightarrow W$ be rational mappings represented by*

$$\phi = \left(\frac{f_1}{g_1}, \ldots, \frac{f_n}{g_n}\right) \quad \text{and} \quad \psi = \left(\frac{f_1'}{g_1'}, \ldots, \frac{f_n'}{g_n'}\right).$$

*Then we say that $\phi = \psi$ if for each $i, 1 \leq i \leq n$, $f_i g_i' - f_i' g_i \in \boldsymbol{I}(V)$.*

The definition makes sense as we need each coordinate of $\phi - \psi = 0$ meaning that each numerator must vanish on $V$. Then, we can now talk about equality of rational mappings.

**Proposition 5.4.** *Two rational mappings $\phi, \psi : V \dashrightarrow W$ are equal if and only if there is a proper subvariety $V' \subseteq V$ such that $\phi$ and $\psi$ are defined on $V \setminus V'$ and $\phi(p) = \psi(p)$ for all $p \in V \setminus V'$.*

Another important operation is rational function composition, which will lead us to the notion of *birational equivalence* and *rational varieties*.

**Definition 5.5** (Composition)**.** *Given $\phi : V \dashrightarrow W$ and $\psi : W \dashrightarrow Z$, we say that $\psi \circ \phi$ is defined if there is a point $p \in V$ such that $\phi$ is defined at $p$ and $\psi$ is defined at $\phi(p)$.*

Let us consider the following exercise when $\phi \circ \psi$ is not well-defined.

**Exercise 5.6.** *The following rational mappings $\phi : \mathbb{R} \dashrightarrow \mathbb{R}^3$ and $\psi : \mathbb{R}^3 \dashrightarrow \mathbb{R}$ is not defined.*

$$\phi(t) = (t, 1/t, t^2) \quad \textit{and} \quad \psi(x, y, z) = \frac{x + yz}{x - yz}.$$

*Proof.* We show that for each $p \in \mathbb{R}$ such that $\phi$ is defined at $p$, $\psi$ is not defined at $\phi(p)$. For each $p \in \mathbb{R} \setminus \{0\}$, we have $\phi(p) = (t, 1/t, t^2)$. Then, we have that

$$\psi \circ \phi(p) = \psi(p, 1/p, p^2) = \frac{p + (1/p)p^2}{p - (1/p)p^2},$$

which is clearly not defined as the denominator is 0. □

**Proposition 5.7.** *Let $\phi : V \dashrightarrow W$ and $\psi : W \dashrightarrow Z$ be rational mappings such that $\psi \circ \phi$ is defined. Then there is a proper subvariety $V' \subsetneq V$ such that:*

(1) *$\phi$ is defined on $V \setminus V'$ and $\psi$ is defined on $\phi(V \setminus V')$.*
(2) *$\psi \circ \phi : V \dashrightarrow Z$ is a rational mapping defined on $V \setminus V'$.*

With the above proposition, we can now define the terminologies we mentioned earlier.

**Definition 5.8.** *We define birational equivalence and rational varieties as follows:*

(1) *Two irreducible varieties $V \subseteq k^m$ and $W \subseteq k^n$ are "birationally equivalent" if there exist rational mappings $\phi : V \dashrightarrow W$ and $\psi : W \dashrightarrow V$ such that $\phi \circ \psi$ is defined (Definition 5.5) and equal to the map $id_W$ (Definition 5.3), and similarly for $\psi \circ \phi$.*
(2) *A "rational variety" is a variety that is birationally equivalent to $k^n$ for some $n$.*

Lastly, there is an analogous theorem to Theorem 4.8, which gives us a method to tell whether two varieties are isomorphic or not through the isomorphism of their coordinate rings. The following theorem can be used to determine whether two varieties are birationally equivalent through the isomorphis of their function fields.

**Theorem 5.9.** *Two irreducible varieties $V$ and $W$ are birationally equivalent if and only if there is an isomorphism of function fields $k(V) \cong k(W)$ which is the identity on $k$. (By definition, two fields are isomorphic if they are isomorphic as commutative rings.)*

Let us exemplify the above definition and theorem via the following exercise:

**Exercise 5.10.** *Show that the singular cubic curve* $V = \boldsymbol{V}(y^2 - x^3)$ *is a rational variety (birationally equivalent to* $k$*).*

*Proof.* We first consider $k[V]$ and will describe $k(V)$ later. By Proposition 3.2, we have that

$$k[V] = k[x,y]/\langle y^2 - x^3\rangle = \{a(x) + b(x)y \mid a, b \in k[x]\},$$

where multiplication is defined by $(a + by)(c + dy) = (ac + bdx^3) + (ad + bc)y$. Considering $V = \mathbf{V}(y^2 - x^3)$, we can parametrize this variety by $y = t^3$ and $x = t^2$, so $V$ is irreducible. This means $k[V]$ is an integral domain. By Definition 5.1, we can now ready to describe $k(V)$.

We claim that $k(V) = k(x) + yk(x)$. For the forward direction, if $c + yd \neq 0 \in k(V)$, then we can write

$$\frac{a + yb}{c + yd} = \frac{a + yb}{c + yd} \cdot \frac{c - yd}{c - yd} = \frac{(ac - bdy^2) + (bc - ad)y}{c^2 - y^2d^2} = \frac{ac - bdx^3}{c^2 - x^3d^2} + y\frac{bc - ad}{c^2 - x^3d^2},$$

which is clearly in $k(x)+yk(x)$. On the other hand, it is clear that any element in $k(x)+yk(x)$ is in $k(V)$.

Now, let us consider the mappings:

$$\alpha : V \to k, (x, y) \mapsto y/x, \quad \text{and} \quad \beta : k \to V, u \mapsto (u^2, u^3),$$

where $\alpha$ is define on the whole $V$ except $(0, 0)$. Then, we can define the pullback mappings

$$\alpha^* : k(u) \to k(V), \quad f(u) \mapsto f(y/x),$$

$$\beta^* : k(V) \to k(u), \quad a(x) + yb(x) \mapsto a(u^2) + u^3b(u^2).$$

We will show that these two mappings $\alpha^*$ and $\beta^*$ define ring isomorphisms between the functional fields $k(V)$ and $k(u)$. We will not show how these two preserve sums and products but how they are inverses. First, let $f(u) \in k(u)$, then we have

$$\beta^*(\alpha^*(f)) = \beta^*(f(y/x)) = f(u^3/u^2) = f(u),$$

so $\beta^* \circ \alpha^*$ us the identity on $k(u)$. On the other hand, for any $a(x) + yb(x) \in k(V)$, we have

$$\alpha^*(\beta^*(a + yb)) = \alpha^*(a(u^2) + u^3b(u^2)) = a(y^2/x^2) + (y^3/x^3)b(y^2/x^2).$$

Since $y^2 = x^3$, we get $y^2/x^2 = x$ and $y^3/x^3 = y$. Hence, $\alpha^*(\beta^*(a + yb)) = a(x) + yb(x)$. Therefore, $\alpha^* \circ \beta^*$ defines the identity on $k(V)$. Hence, $k(V) \cong k(u)$. By theorem 5.9, we conclude that $V$ and $k$ are birationally equivalent, or that $V$ is a rational variety. ∎

Note that even though two varieties are not isomorphic, they can have the same function field, so they are birationally equivalent. This implies that birational equivalence of irreducible varieties is a weaker equivalent relation than isomorphism. However, the classification of irreducible varieties up to birational equivalence turns out to be a more attractive research topic than the one up to isomorphism. This is because birational equivalence is usually easier to construct.

While there are many more interesting things to talk about in §6 of this chapter. I think these 5 sections have given us sufficient knowledge of rational equivalence and variety isomorphism, which are the main objects we want to study in this project.

## References

[1] D. Cox, J. Little, and D. Oshea, *Ideals, varieties, and algorithms: an introduction to computational algebraic geometry and commutative algebra.* Springer Science & Business Media, 2013.

(Nawapan Wattanawanichkul) Bowdoin Colleg,Brunswick, ME 04011
*Email address*: nwattana@bowdoin.edu