HYPERBOLIC TRIGONOMETRY: DEVELOPMENT, IDENTITIES, AND APPLICATIONS

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Abstract. While the study of trigonometry stemmed from the investigation of triangles in the past, the development of "hyperbolic" trigonometry is different and interesting. This paper summarizes the history of hyperbolic trigonometry and presents several remarkable hyperbolic trigonometry identities that have nice Euclidean counterparts. Lastly, we provide a couple of applications to emphasize the importance of this topic that is not limited to theoretical mathematics.

Contents

1. Definitions and History

While hyperbolic trigonometry is the main focus of this paper, the study of this topic is indeed originated from the study of the regular trigonometry, or what we call the "circular" trigonometry. Then, it is important to recall what the circular trigonometry is so that we can compare the two types of trigonometries.

Date: August 2, 2021.

Key words and phrases. Hyperbolic trigonometry, circular trigonometry, the catenary curve, Poincaré distance, Bolyai-Lobachevsky formula.

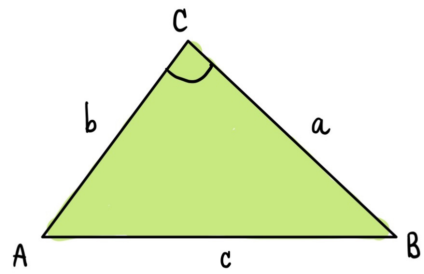

FIGURE 1. A problem of finding the length c given a, b and angle C.

1.1. **Circular Trigonometry.** The study of circular trigonometry started from the study of triangles. A problem that mathematicians in the ancient time studied was finding the length of the remaining side of triangle ABC, given we know sides a, b and the angle C as shown in Figure 1. This kind of question then gave rise to trigonometry notions for right triangles, which is the type of trigonometry we have learned since high school. The followings are the main functions in trigonometry.

Definition 1.1. *For any right triangle ABC in the Euclidean geometry with the length of the side opposite to the angles $A, B,$ and C denoted as $a, b,$ and c and C is the right angle, the functions* $\sin A, \cos A,$ *and* $\tan A$ *can be defined as*

$$\sin A = \frac{a}{c}, \ \cos A = \frac{b}{c}, \ \ \textit{and} \ \ \tan A = \frac{a}{b}.$$

Furthermore, we also define

$$\csc A = \frac{1}{\sin A}, \ \sec A = \frac{1}{\cos A}, \ \ \textit{and} \ \ \cot A = \frac{1}{\tan A}.$$

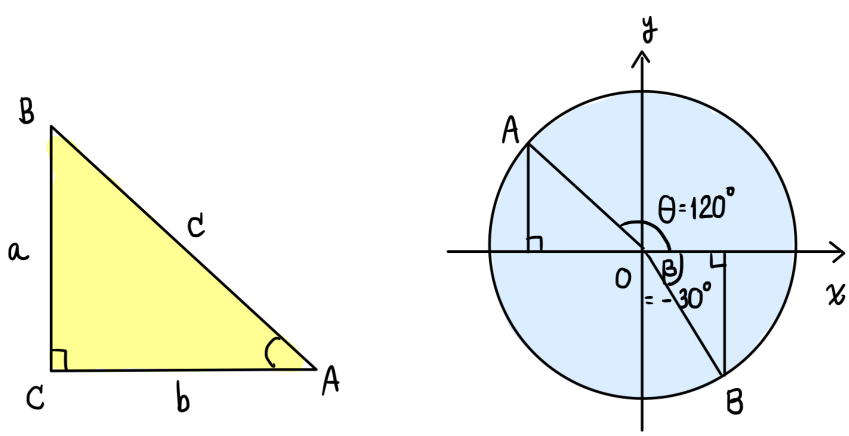

FIGURE 2. The right triangle model (left) and the circular model (right).

With Definition 1.1, mathematicians established a great collection of useful identities that apply to both right triangles and any triangles in the Euclidean geometry, such as the law of sines and the law of cosines. The example shown in Figure 1 can, indeed, be solved by using the law of cosines.

Notice that Definition 1.1 only allow the domain of functions sin and cos to be positive and less than π. Thus, mathematicians use the unit circle $x^2 + y^2 = 1$ to assist in extending the domain of both functions, as shown in Figure 2. Turns out, we found the coordinates $(\cos\theta, \sin\theta)$ for any real number θ lie on the unit circle. From now on, we will refer to this regular trigonometry as circular trigonometry to distinguish it from the hyperbolic one.

For the modern definitions, we usually refer to $\sin x$ and $\cos x$ by their Taylor series expansions:

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots,$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots.$$

With some knowledge in complex analysis, namely, the Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$, one can easily verify that

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \text{ and } \cos x = \frac{e^{ix} + e^{-ix}}{2}. \tag{1.1}$$

1.2. **Hyperbolic Trigonometry.** Unlike the study of circular trigonometry, the origin of hyperbolic trigonometry did not start from the investigation of hyperbolic triangles, but the following two functions.

Definition 1.2. *Let* sinh *be the hyperbolic sine function and* cosh *the hyperbolic cosine function, then* $\sinh x$ *and* $\cosh x$ *can be defined as follow.*

$$\sinh x := \frac{e^x - e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots,$$

$$\cosh x := \frac{e^x + e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots.$$

According to Barnett [B], these two functions that define sinh and cosh,

$$\frac{e^x - e^{-x}}{2} \text{ and } \frac{e^x + e^{-x}}{2}, \tag{1.2}$$

have been studied since the late 17th century when scholars tried to figure out the model of *catenary* curve. (The problem of catenary curve will be discussed later in this paper.) However, a mathematician that arguably made a huge contribution to the study of hyperbolic trigonometry is *Johann Heinrich Lambert*, a Swiss polymath. Lambert published one of his study on sin and cos functions in 1761, and, in the same paper, he further studied the analogues of sin and cos which are those two functions in (1.2). At that time, he did not give any special names to the two functions, but he already noticed that the functions are tightly related to the unit hyperbola $x^2 - y^2 = 1$.

In fact, around $1757 - 1762$, *Jacopo Riccati*, a Venetian mathematician, had also discovered that these two functions are connected to the unit hyperbola and named them "hyperbolic sine" and "hyperbolic cosine," respectively. Unfortunately, Riccati's work is not as frequently mentioned as the one from Lambert due to its difficult languages and notations.

1.3. **The Unit Circle and the Unit Hyperbola.** As mentioned in the previous subsection, Lambert and Riccati noticed a nice connection between sinh, cosh and the unit hyperbola. So we would like to note Lambert's observation here.

Lemma 1.3. *The coordinates* $(\cosh\theta, \sinh\theta)$ *parametrize the positive unit hyperbola* $x^2 - y^2 = 1$*, (i.e., the branch in the right half plane.)*

Proof. Consider the following computation,

$$\begin{aligned}\cosh^2\theta - \sinh^2\theta &= \left(\frac{e^\theta + e^{-\theta}}{2}\right)^2 - \left(\frac{e^\theta - e^{-\theta}}{2}\right)^2 \\ &= \frac{e^{2\theta} + 2 + e^{-2\theta}}{4} - \frac{e^{2\theta} - 2 + e^{-2\theta}}{4} \\ &= \frac{4}{4} = 1. \end{aligned} \tag{1.3}$$

Notice that $\cosh\theta$ can be any real number in the range $[1, \infty)$, where $\cosh 0 = 1$. Thus, (1.3) implies that any point on the positive unit hyperbola $x^2 - y^2 = 1$ can be parametrized by $(\cosh\theta, \sin\theta)$ as desired. ☐

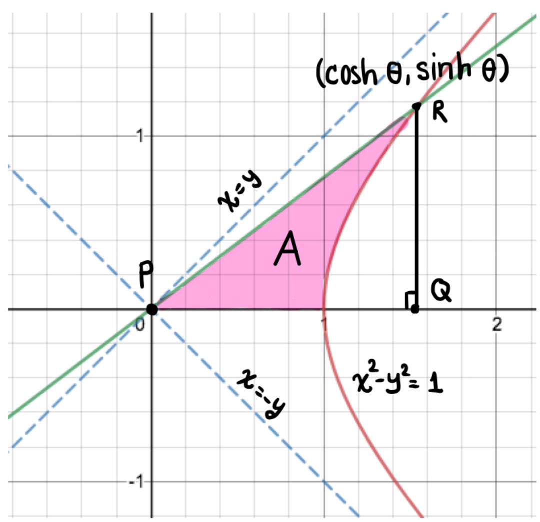

FIGURE 3. The area bounded by the curve of the unit hyperbola and the line passing the origin and $(\cosh\theta, \sinh\theta)$.

While this parametrization works out algebraically, one question that could arise is what θ represents in this geometric view. Clearly, as there exist two asymptotes that bound the unit hyperbola, namely, the lines $x = y$ and $x = -y$, if θ were the angle measured from the positive $x-$axis, the value of θ for sinh and cosh would be limited to some range of real number. However, by Definition 1.2, θ can be any real number, so its geometric meaning must be something else.

Lemma 1.4. *In fact, θ, the argument for* sinh *and* cosh, *represents twice of the area bounded by the positive unit hyperbola, the $x-$axis, and the line passing the origin and* $(\cosh\theta, \sinh\theta)$.

Proof. As in Figure 3, we can compute area A as follows

$$\begin{aligned}[\text{area } A] &= [\Delta PQR] - [\text{area under the curve}] \\ &= \frac{\sinh\theta \cdot \cosh\theta}{2} - \int_{x=\cosh 0}^{\cosh\theta} \sqrt{x^2-1}\, dx \\ &= \frac{\sinh\theta \cdot \cosh\theta}{2} - \int_0^\theta \sinh^2\theta\, d\theta \\ &= \frac{\sinh\theta \cdot \cosh\theta}{2} - \int_0^\theta \frac{e^{2\theta} - 2 + e^{-2\theta}}{4}\, d\theta \\ &= \frac{\sinh 2\theta}{4} - \frac{\sinh 2\theta - 2\theta}{4} = \frac{\theta}{2}.\end{aligned}$$

Thus, θ is twice the area A as desired. □

While the fact that θ is proportionate to the area bounded by the unit hyperbola is not very intuitive, it is not totally out of place as the analogue of Lemma 1.4 also appears in the circular model.

Lemma 1.5. *The argument for* sin *and* cos θ, *represents twice of the sector area bounded by the unit circle, the $x-$axis, and the line passing the origin and* $(\cos\theta, \sin\theta)$.

Proof. Recall that the area of the unit circle is $\pi \cdot 1^2 = \pi$ and the angle around the origin is 2π. Thus, the area of the sector is clearly

$$\frac{\theta}{2\pi} \cdot \pi = \frac{\theta}{2}.$$

□

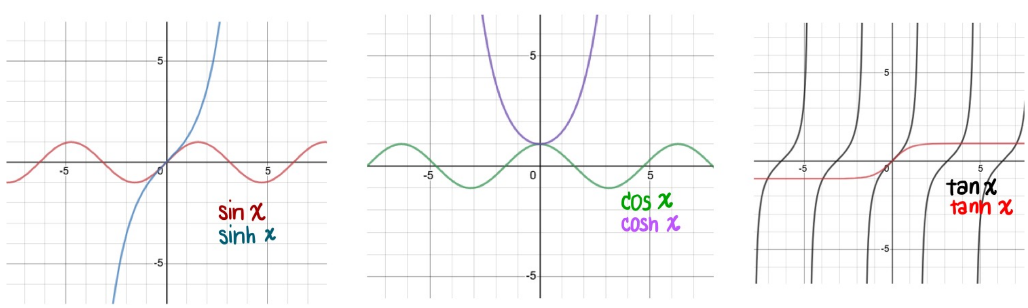

FIGURE 4. Comparison between circular trigonometry and Hyperbolic trigonometry

In conclusion, the development of hyperbolic trigonometry is somehow opposite to that of circular trigonometry. It started from the function (1.2) involving e.

Then, mathematicians connected these two functions to the positive unit hyperbola and later used them to solve problems related to hyperbolic triangles as we will see in §3. Before we move on to the next section, it is also worth noting that the functions $\sinh, \cosh$ and $\tanh$ are not periodic as shown in Figure 4.

2. Relationship between Circular and Hyperbolic Trigonometries

Now that we know the definitions of both trigonometries, it is natural to ask whether there is any connection between them. However, it requires some tools to help us understand their relationship, namely, the Poincaré distance and the Bolyai-Lobachevsky Formula.

2.1. Poincaré Distance.

First of all, we refer the following useful definition.

Definition 2.1. *If P, Q, A, and B are distinct points in $\mathbb{R}^2$, then their* cross-ratio *is*

$$[P, Q, A, B] = \frac{PB \cdot QA}{PA \cdot QB},$$

where PB, QA, PA, and B are the Euclidean lengths of those segments.

Then, we can use the cross ratio of four distinct points to find the Poincaré length of a line segment.

Definition 2.2. *If P, Q, A, and B are distinct points in $\mathbb{R}^2$, then in hyperbolic geometry, the Poincaré length $d(A, B)$ is defined as*

$$d(A, B) = |\ln([P, Q, A, B])|.$$

In particular, using Definition 2.1 and 2.2, we can find the Poincaré distance from the origin to any point in the disk.

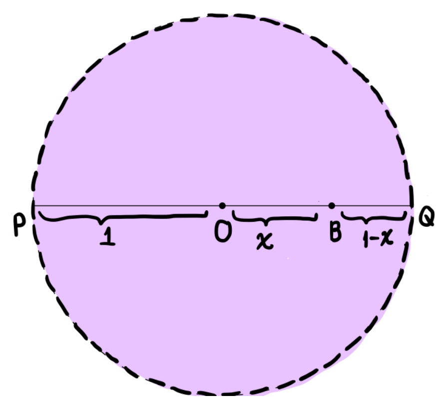

Figure 5. Poincaré distance from the origin.

Theorem 2.3. *If a point B inside the unit disk is at a Euclidean distance x from the origin O, then the Poincaré distance from B to O is given by*

$$d(B, O) = \left|\ln\left(\frac{1+x}{1-x}\right)\right|.$$

Proof. This theorem can be easily proved by considering Figure 5. We set PQ to be the diameter of the disk. We then know that the Euclidean distances of $PO = OQ = 1$ and $BQ = 1 - x$. Thus, by Definition 2.1, we have

$$[P, Q, O, B] = \frac{PB \cdot QO}{PO \cdot QB} = \frac{1+x}{1-x}.$$

Then, by Definition 2.2, we obtain the desired result. □

2.2. **Bolyai-Lobachevsky Formula.** Before we talk about the remarkable formula of Bolyai-Lobachevsky, let us recall some basic knowledge in the hyperbolic geometry, namely, the angle of parallelism.

Recall that the axiom that distinguishes the Euclidean and hyperbolic geometry is the fifth axiom regarding parallelism. For the Euclidean geometry, the axiom states that for every line l and for every point P that does not line on l, there exists a unique line m through P that is parallel to l. On the other hand, for the hyperbolic geometry, there exist more than one line M that passes through P and parallel to l. Thus, the existence of more than one parallel lines in hyperbolic geometry gives rise to the notions of *limiting parallel rays* and *the angle of parallelism.*

Theorem 2.4. *Given any line l and any point P not on l, there exist limiting parallel rays $\overrightarrow{\mathrm{PX}}$ and $\overrightarrow{\mathrm{PY}}$. Furthermore, the angle between the limiting rays and the line through P perpendicular to l is called the angle of parallelism, which is denoted as α in Figure 6.*

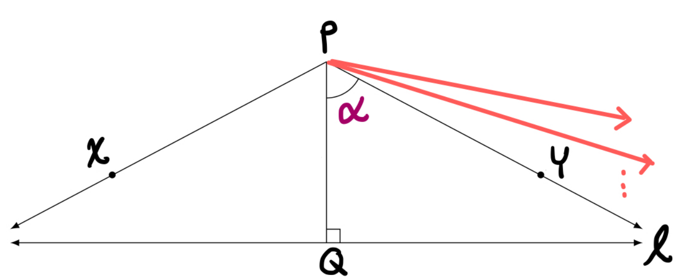

FIGURE 6. The angle of parallelism.

The Bolyai-Lobachevsky formula is then a remarkable formula that connects the angle of parallelism with the Hyperbolic distance from P to Q.

Theorem 2.5. *Let α be the angle of parallelism for P with respect to l and d be the hyperbolic distance from P to Q, where PQ is perpendicular to l. We then have, the formula of Bolyai-Lobachevsky:*

$$\tan\left(\frac{\alpha}{2}\right) = e^{-d}.$$

Proof. We first draw a tangent line of the circular arc at point P intersect QR at S. In Figure 7, we denote x as the euclidean distance from P to Q, thus we have, by Theorem 2.3,

$$d = \left| \ln\left(\frac{1+x}{1-x}\right) \right| \implies e^{-d} = \left(\frac{1-x}{1+x}\right). \tag{2.1}$$

Now, considering $\triangle PSR$, since PS and SR are two tangent lines intersecting at S, $PS \cong SR$. Thus, $\angle SPR = \angle PRS = \beta$ and $\angle PSR = 2\beta$. Considering $\triangle PQS$, we then have $\alpha + 2\beta = \pi/2$, or $\alpha/2 = \pi/4 - \beta$.

We then apply the tangent rule to $\alpha/2 = \pi/4 - \beta$.

$$\begin{aligned}\tan(\alpha/2) &= \tan(\pi/4 - \beta) \\ &= \frac{\tan(\pi/4) - \tan(\beta)}{\tan(\pi/4) + \tan(\beta)} \\ &= \frac{1 - \tan(\beta)}{1 + \tan(\beta)}.\end{aligned}$$

To finish up the proof, we notice that $\tan(\beta)$ is indeed $PQ/QR = x/1 = x$. Combining this fact with (2.1), yields the result. $\square$

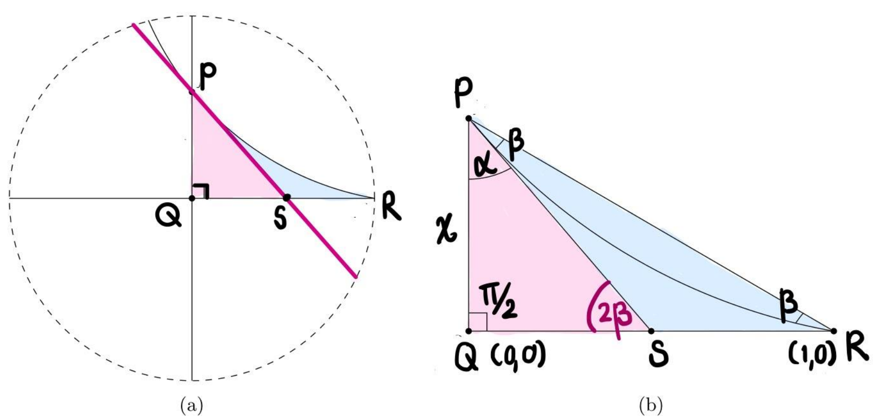

FIGURE 7. Proof of Bolyai-Lobachevsky formula.

Theorem 2.6. *Lobachevsky denoted α or the angle of parallelism as $\Pi(d)$ as α is dependent on d. Then, an alternative form of the Bolyai-Lobachevsky formula can be written as*

$$\Pi(d) = 2\arctan(e^{-d}),$$

which provides the radian measure of the angle of parallelism.

This alternative form gives us a clear link between hyperbolic and circular functions, and, by manipulating the Bolyai-Lobachevsky formula, we obtain the following formulas.

Theorem 2.7. *Let $\Pi(x)$ be the angle of parallelism and x be the hyperbolic distance. Then,*

$$\sin(\Pi(x)) = \operatorname{sech}(x) = 1/\cosh(x), \tag{2.2}$$

$$\cos(\Pi(x)) = \tanh(x), \tag{2.3}$$

$$\tan(\Pi(x)) = \operatorname{csch}(x) = 1/\sinh(x). \tag{2.4}$$

Proof. Since the proof for (2.2) and (2.3) can be done similarly and (2.4) follows quite immediately, we then present only the proof of the first formula here.

Let $y = \arctan(e^{-d})$, then we have $\tan(y) = e^{-d}$. Thus, $\sec^2(y) = \tan^2(y) + 1$ is equivalent to $\sec^2(y) = e^{-2x} + 1$.

Then, we have

$$\cos(y) = \frac{1}{(\sec^2(y))^{1/2}} = \frac{1}{(e^{-2x}+1)^{1/2}}.$$

Similarly, we also have

$$\sin(y) = \tan(y)\cos(y) = \frac{e^{-x}}{(e^{-2x}+1)^{1/2}}.$$

Therefore, by the double angle formula, we obtain

$$\sin(\Pi(d)) = \sin(2y) = 2\sin(y)\cos(y) = \frac{e^{-x}}{e^{-2x}+1} = \frac{1}{e^{x}+e^{-x}} = \operatorname{sech}(x).$$

□

As we can see, with some help of cross-ratio and the Bolyai-Lobachevsky formula, we succeed in developing a nice connection between the circular trigonometry and hyperbolic trigonometry functions.

3. Hyperbolic Trigonometry Identities

The name "hyperbolic trigonometry" does not only refer to the fact that coordinates $(\cosh\theta, \sinh\theta)$ fit nicely on the unit hyperbola, but also that hyperbolic trigonometry functions, i.e., $\sinh, \cosh, \tanh$, allow us to develop useful identities in hyperbolic triangles. Just like the circular trigonometry identities, some hyperbolic trig identities only hold for right hyperbolic triangles while others work out well for any hyperbolic triangles.

3.1. **Identities on Right Triangles.**

Let us first introduce the most three common identities that hold for hyperbolic right triangles.

Theorem 3.1. *Given any triangle $\triangle ABC$, with $\angle C$ being the right angle, in the hyperbolic plane. Let a, b, and c denote the hyperbolic lengths of the corresponding sides. Then*

$$\sin A = \frac{\sinh a}{\sinh c} \quad \textit{and} \quad \cos A = \frac{\tanh b}{\tanh c}, \tag{3.1}$$

$$\cosh c = \cosh a \cdot \cosh b = \cot A \cdot \cot B, \tag{3.2}$$

$$\cosh a = \frac{\cos A}{\sin B}. \tag{3.3}$$

While the proofs of these formulas are interesting and involving constructing some figure in the Poincaré model, their Euclidean counterparts are even more worth noting. In this section, we decided to focus mainly on simplifying these identities under a certain set of assumptions.

From the first part of (3.2), we can replace $\cosh a, \cosh b$, and $\cosh c$ by their Taylor series expansions and obtain

$$\begin{aligned}\cosh c &= \cosh a \cdot \cosh b \\ \sum_{n=0}^{\infty} \frac{c^{2n}}{(2n)!} &= \sum_{n=0}^{\infty} \frac{a^{2n}}{(2n)!} \cdot \sum_{n=0}^{\infty} \frac{b^{2n}}{(2n)!} \\ 1 + \frac{1}{2}c^2 + \frac{1}{4!}c^4 + \cdots &= 1 + \frac{1}{2}(a^2 + b^2) + \frac{1}{4!}(a^4 + 6a^2b^2 + b^4) + \cdots .\end{aligned}$$

When we consider the last equation under the assumption that triangle ABC is sufficiently small, the higher order terms can be ignored. This is because with small a, b and c, for large n a^n, b^n, and c^n grow much more slowly compared with $(2n)!$. Thus, it leaves us with

$$c^2 \approx a^2 + b^2,$$

which resembles the Pythagoras theorem in the Euclidean geometry.

Similarly, for (3.1), under the same assumption, we also have that

$$\sin A \approx \frac{a}{c} \quad \text{and} \quad \cos A \approx \frac{b}{c},$$

which are just the trigonometry identities on any right triangle in the Euclidean geometry.

	Hyperbolic	Euclidean	Difference
$a = 4, b = 1$	$c = 4.45$	$c = 4.12$	.33
$a = 5, b = 2$	$c = 6.33$	$c = 5.36$	.97
$a = 6, b = 3$	$c = 8.31$	$c = 6.71$	1.6
$a = 13, b = 10$	$c = 22.3$	$c = 16.4$	5.9
$a = 25, b = 30$	$c = 54$	$c = 39.05$	14.95
$a = 40, b = 50$	$c = 89.3$	$c = 64$	25.3

Table 1. Comparing the length of c in both geometries.

Table 1 clarifies what it means to be sufficiently small for a hyperbolic triangle ABC. In particular, when we find the length of c in the hyperbolic geometry, we require Formula (2.3) while, for the Euclidean geometry, we require the Pythagoras theorem. Also, note that not every hyperbolic identities have nice Euclidean counterparts. Here, there are no nice Euclidean counterparts for the second part of (3.2) and (3.3).

3.2. **Identities on Any triangles.** Now, let us introduce more identities that work nicely on any hyperbolic triangles.

Theorem 3.2. *For any triangle $\triangle ABC$ in the hyperbolic plane,*

$$\frac{\sin A}{\sinh a} = \frac{\sin B}{\sinh b} = \frac{\sin C}{\sinh c}, \tag{3.4}$$

$$\cosh c = \cosh a \cdot \cosh b - \sinh a \cdot \sinh b \cdot \cos C, \tag{3.5}$$

$$\cosh c = \frac{\cos A \cdot \cos B + \cos C}{\sin A \cdot \sin B}. \tag{3.6}$$

Similarly, under the assumption of a sufficiently small triangle, (3.4) can be reduced to the law of sines and (3.5) can be reduced to the law of cosines in the Euclidean geometry.

In particular, for (3.5), we can replace $\cosh a, \cosh b$, and $\cosh c$ by their Taylor series expansions and obtain

$$\begin{aligned} \cosh c &= \cosh a \cdot \cosh b - \sinh a \cdot \sinh b \cdot \cos C \\ \sum_{n=0}^{\infty} \frac{c^{2n}}{(2n)!} &= \sum_{n=0}^{\infty} \frac{a^{2n}}{(2n)!} \cdot \sum_{n=0}^{\infty} \frac{b^{2n}}{(2n)!} - \sum_{n=0}^{\infty} \frac{a^{2n+1}}{(2n+1)!} \cdot \sum_{n=0}^{\infty} \frac{b^{2n+1}}{(2n+1)!} \cdot \cos C \\ 1 + \frac{1}{2}c^2 + \cdots &= \left(1 + \frac{1}{2}(a^2 + b^2) + \cdots\right) - (ab + \cdots) \cdot \cos C. \end{aligned}$$

Again, the higher order terms can be ignored under the assumption, giving us

$$c^2 \approx a^2 + b^2 - 2ab \cdot \cos C,$$

which resembles the law of cosines in the Euclidean geometry.

For (3.4), it is quite clear that those denominators $\sinh a, \sinh b$, and $\sinh c$ can be reduced to a, b, c implying that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c},$$

which is the law of sines. Lastly, the formula (3.6), despite its usefulness, does not have a nice Euclidean analogue.

3.3. **Comparison and Connection.** Before we shift our focus to the applications of hyperbolic trigonometry identities, let us summarize all the identities and interesting facts we have discussed in the following table.

	Hyperbolic	**Euclidean**
θ	twice the area bounded by the curve	twice the sector area
Pythagoras	$\cosh c = \cosh a \cdot \cosh b$	$c^2 = a^2 + b^2$
Right $\triangle$ defs	$\sin A = \frac{\sinh a}{\sinh c},\ \cos A = \frac{\tanh b}{\tanh c}$	$\sin A = \frac{a}{c},\ \cos A = \frac{b}{c}$
Law of sines	$\frac{\sin A}{\sinh a} = \frac{\sin B}{\sinh b} = \frac{\sin C}{\sinh c}$	$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$
Law of cosines	$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos C$	$c^2 = a^2 + b^2 - 2ab \cos C$

TABLE 2. Comparing hyperbolic trig identities and their Euclidean counterparts.

4. Applications

When asking about the applications of hyperbolic trigonometry, the most common answer always involves a catenary curve, an old physics problem first solved in the 17th century.

4.1. **The Catenary Curve.**

The catenary curve refers to the shape of a flexible inextensible cord that is hung freely from the two fixed point. The word "caternary" was first used by Huygens in 1690, yet the study of the shape of such a curve had been examined as early as the 15th century by da Vinci. One of the most notable scientists like Galileo believed that the curve is parabola, but the claim was disproved in the 17th century.

The catenary problem was widely discussed right after Jakob Bernoulli posed it as a challenge in a 1690 *Acta Eruditorum* paper. Then, in June 1691, there appeared three independent solutions given by Christian Huygens, Gottfried Leibniz and Johann Bernoulli.

Theorem 4.1. *The catenary curve can be described by the equation*

$$y = \frac{e^{ax} + e^{-ax}}{2a} = \frac{cosh(ax)}{a},$$

where a is some constant depending on the cord.

As a matter of fact, at that time, the three solutions did not mention hyperbolic functions or any other explicit function, but merely curve constructions. Hence, while the study of catenary is connected to the hyperbolic trigonometry, it does not serve as where the study of the hyperbolic trigonometry actually began.

In this paper, we present a modern proof of 4.1 from [P] which is much more easier than those offered by the three great mathematicians, thanks to the innovation of differential equations and hyperbolic trigonometry.

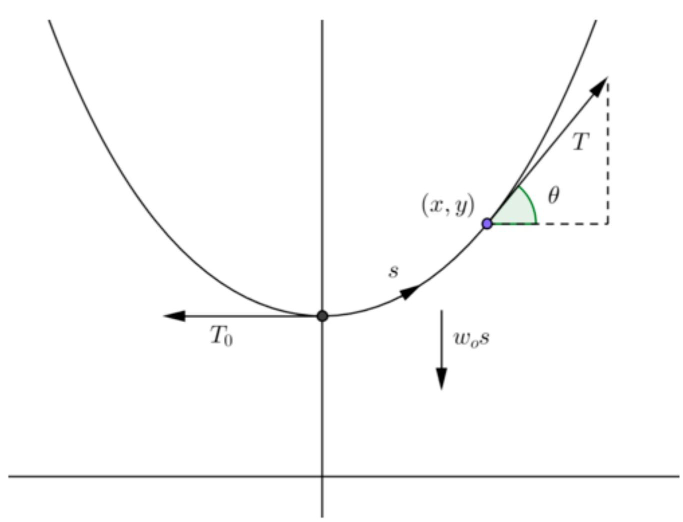

Figure 8. The set up for Theorem 4.1.

Proof. Let (x, y) be an arbitrary point on the cord, s the length along the arc of the cord from the lowest point to (x, y), and w_o the linear density of the cord, i.e., its weight per unit length.

Since the cord is assumed to be ideally flexible inextensible one, the tension T is along the cord, thus along the tangent at any point when freely hung down. Then, given T_0, the tension at the lowest point, T, the tension at the point (x, y), and $w_0 s$, the weight between the two points, we have

$$\begin{aligned} T_0 &= T\cos\theta, \\ w_0 s &= T\sin\theta. \end{aligned}$$

Dividing these two equations and setting $a = w_0/T$ give us $\tan\theta = as = \dfrac{dy}{dx}$.

Differentiating with respect to x, along with using the derivative of arc length, yields

$$\frac{d^2y}{dx^2} = a\frac{ds}{dx} = a\sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \tag{4.1}$$

Then, substituting $p = dy/dx$ transforms (4.1) into $dp/dx = a\sqrt{1+p^2}$, which can be solved by separation of variables

$$\int \frac{dp}{\sqrt{1+p^2}} = \int a dx. \tag{4.2}$$

The by solving the left and right hand side integral of (4.2), we obtain

$$\ln(\sqrt{1+p^2} + p) = ax + c_3 = a_x,$$

as when $x = 0$, c_3 can be solved to 0.

By doing some algebra, this gives us

$$p = \frac{dy}{dx} = \frac{e^{ax} - e^{-ax}}{2}.$$

Lastly, by derivative of exponential function, we obtain the desired result

$$y = \frac{e^{ax} + e^{-ax}}{2a} + c_4,$$

where c_4 can be solved to 0 when $x = 0$. □

4.2. **More applications.** The catenary curve does not only appear in an ideal cord, but also in architecture. Imagine that you have a free-standing arch, meaning no outside supports, then the optimal shape to handle the lines of thrust produced by its own weight is $\cosh(x)$. An example that is widely known is the dome of Saint Paul's Cathedral in England (see Figure 9), which has a $\cosh(x)$ cross-section. Antoni Gaudí, a Catalan architect, also exploits this kind of arch in his work.

FIGURE 9. The dome of Saint Paul's Cathedral in England.

Another place that the catenary curve appears is its corresponding surface of revolution, the catenoid. This surface can be understood as the form that a soap bubble takes when it is stretched across two rings as shown in Figure 10.

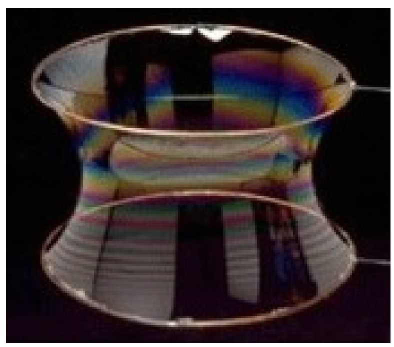

FIGURE 10. The Catenoid bubble.

Going beyond the catenary curve, the hyperbolic trigonometry functions are widely used in mathematics and physics such as in the Mercator projection, a technique used to create a map, and in velocity addition in special relativity, etc.

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